

# A solution space for a system of null-state partial differential equations I

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In this article, we study a system of  $2N + 3$  linear homogeneous second-order partial differential equations (PDEs) in  $2N$  variables that arise in conformal field theory (CFT) and multiple Schramm-Löwner Evolution (SLE $_{\kappa}$ ). In CFT, these are null-state equations and Ward identities. They are satisfied by partition functions central to the characterization of a statistical cluster or loop model such as percolation, or more generally the Potts models and  $O(n)$  models, at the statistical mechanical critical point in the continuum limit. (Certain multiple-SLE $_{\kappa}$  partition functions also satisfy these equations.) The partition functions for critical lattice models contained in a polygon  $\mathcal{P}$  with  $2N$  sides exhibiting free/fixed side-alternating boundary conditions are proportional to the CFT correlation function

$$\langle \psi_1^c(w_1) \psi_1^c(w_2) \dots \psi_1^c(w_{2N-1}) \psi_1^c(w_{2N}) \rangle_{\mathcal{P}},$$

where the  $w_i$  are the vertices of  $\mathcal{P}$  and  $\psi_1^c$  is a one-leg corner operator. Partition functions conditioned on crossing events in which clusters join the fixed sides of  $\mathcal{P}$  in some specified connectivity are also proportional to this correlation function. When conformally mapped onto the upper half-plane, methods of CFT show that this correlation function satisfies the system of PDEs that we consider.

This article is the first of two papers in which we completely characterize the space of all solutions for this system of PDEs that grow no faster than a power-law. In this first article, we use methods of analysis to prove, to within a precise technical conjecture, that the dimension of this solution space is no more than  $C_N$ , the  $N$ th Catalan number. In the appendices, we propose a method for proving the conjecture mentioned, and we posit that all classical solutions for this system of PDEs indeed grow no faster than a power law. In the second article, we use the results herein to prove that the solution space has dimension  $C_N$  and is spanned by solutions constructed with the CFT Coulomb gas (contour integral) formalism.

Keywords: conformal field theory, Shramm-Löwner Evolution

## I. INTRODUCTION

We consider critical bond percolation on a very fine square lattice inside a rectangle  $\mathcal{R} := \{x + iy \mid 0 < x < R, 0 < y < 1\}$  with *wired* (or *fixed*) left and right sides (i.e., all bonds are activated on these sides) and *free* top and bottom sides (i.e., we do not condition any of the bonds on these sides). In [1], J. Cardy used conformal field theory (CFT) [2–4] methods to argue that, at the critical point and in the continuum limit, the partition function for this system is proportional to the CFT correlation function

$$\langle \psi_1^c(w_1) \psi_1^c(w_2) \psi_1^c(w_3) \psi_1^c(w_4) \rangle_{\mathcal{R}}, \quad (1)$$

where the  $w_i$  are the vertices of  $\mathcal{R}$  and  $\psi_1^c(w_i)$  is a  $c = 0$  CFT one-leg corner operator [5] that implements the boundary condition change from free to fixed [1, 6, 7]. The essence of this argument supposed the emergence of conformal invariance at the critical point for bond percolation in the continuum limit, a feature that was previously

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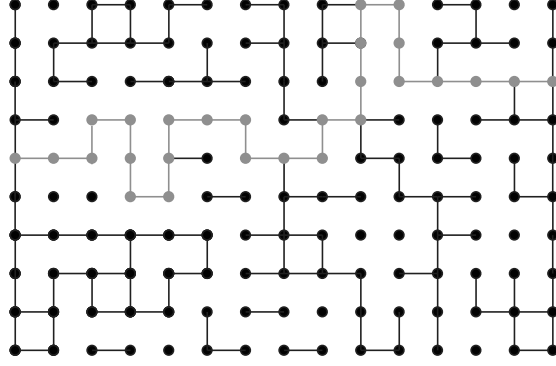


FIG. 1: An illustration of a bond percolation configuration in a rectangle with a crossing (gray bonds) from the left side of the rectangle to the right side.

observed in computer simulations [8]. By treating bond percolation as the  $Q \rightarrow 1$  limit of the random cluster model, Cardy further argued that the *crossing probability*, or the probability that the wired sides of  $\mathcal{R}$  are joined by a cluster of activated bonds (figure 1), is proportional to this correlation function too. This argument, together with methods of CFT, led to an explicit prediction for the crossing probability as a function of the rectangle's aspect ratio  $R$  [1]:

$$\mathbb{P}\{\text{left-right crossing}\} = \frac{3\Gamma(2/3)}{\Gamma(1/3)^2} \eta^{1/3} {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{4}{3} \middle| \eta\right), \quad R = K(1-\eta)/K(\eta). \quad (2)$$

Here,  $\eta \in (0, 1)$  corresponds one-to-one with the aspect ratio  $R \in (0, \infty)$  of  $\mathcal{R}$  via the second equation in (2), with  $K$  the complete elliptic function of the first kind. The prediction (2), called *Cardy's formula*, was numerically verified by high-precision computer simulations [1, 8], thus affirming the presence of conformal symmetry in the continuum limit of critical percolation. Further simulations [9] consistently suggest that many observables, such as the probability of the left-right cluster crossing event, common to different homogeneous models of critical percolation (e.g., site vs. bond percolation, percolation on different lattices) converge to the same value in the continuum limit, a phenomenon called *universality*. Later, S. Smirnov rigorously proved Cardy's formula for site percolation on the triangular lattice [10].

The setup for Cardy's formula has interesting generalizations that motivate the analysis presented in this article. Looking beyond rectangles, we may consider system domains that are even-sided polygons  $\mathcal{P}$ , with the boundary condition (BC) alternating from wired to free to wired, etc., as we trace the boundary of  $\mathcal{P}$ . We call this a free/fixed side-alternating boundary condition (FFBC) (figure 2). And looking beyond percolation, we may consider other lattice models with critical points that have CFT descriptions in the continuum limit. These include the Potts model and its close relative, the random cluster model. If we condition these systems so they exhibit a specified FFBC event  $\varsigma$  on the boundary of  $\mathcal{P}$ , then Cardy's argument may be adapted to imply that the conditioned partition function is

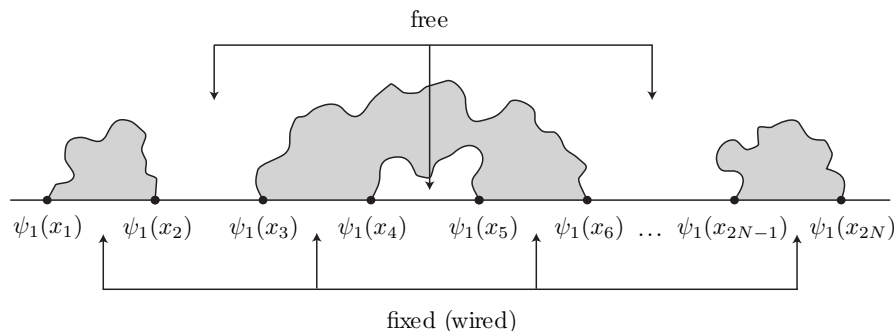


FIG. 2: An illustration of a sample in an FFBC event. Boundary clusters anchor to the fixed, or wired, segments of the real axis, and their perimeters, the boundary arcs, anchor to the BCCs. In this figure, we have conformally mapped the interior of the polygon onto the upper half-plane, with  $x_i$  the image of the  $i$ th vertex.

proportional to the  $2N$ -point function

$$\langle \psi_1^c(w_1) \psi_1^c(w_2) \dots \psi_1^c(w_{2N-1}) \psi_1^c(w_{2N}) \rangle_{\mathcal{P}}^{\varsigma}. \quad (3)$$

Here, the  $w_i$  are the vertices of  $\mathcal{P}$ , and  $\psi_1^c(w_i)$  is a CFT one-leg corner operator [5] that implements the boundary condition change from free to fixed [1, 6, 7]. (More about corner operators can be found in [6, 11, 12], and appendix A surveys some aspects of the application of CFT to critical lattice models that are essential to this article.) If we condition the system on a particular FFBC event  $\varsigma$ , then Potts model spin-clusters or FK-clusters, either are called *boundary clusters*, anchor to the wired sides of  $\mathcal{P}$  and join these sides in some topological crossing configuration with some non-trivial probability. An induction argument [13] shows that there are  $C_N$  such configurations (figure 3), with  $C_N$  the  $N$ th Catalan number given by

$$C_N = \frac{(2N)!}{N!(N+1)!}. \quad (4)$$

*Crossing formulas* that give the probability of these crossing events as a function of the shape of  $\mathcal{P}$  generalize Cardy's formula, which corresponds to the case of critical percolation with  $N = 2$ . The analysis in this article and its sequel [14] delivers a new method for calculating these crossing formulas that we exploit in [15]. This method extends recent results on crossing probabilities for hexagons [16, 17].

To calculate the  $2N$ -point function (3), we conformally map  $\mathcal{P}$  onto the upper half-plane (figure 2). After we continuously extend it to the boundary of  $\mathcal{P}$ , this map also sends the vertices  $w_1, w_2, \dots, w_{2N}$  onto real numbers  $x_1 < x_2 < \dots < x_{2N}$ , and it sends the one-leg corner operator  $\psi_1^c(w_i)$  hosted by the  $i$ th vertex of  $\mathcal{P}$  to a one-leg boundary operator  $\psi_1(x_i)$  [5]. In the Potts model (resp. random cluster model) [18, 19], the one-leg boundary operator is a primary operator that belongs to the  $(2, 1)$  (resp.  $(1, 2)$ ) position of the Kac table [1, 7]. (One-leg boundary operators are discussed in further detail in appendix A.) The CFT null-state condition implies that the half-plane  $2N$ -point function satisfies the system of  $2N$  *null-state partial differential equations (PDEs)* [2, 3]

$$\left[ \frac{3\partial_j^2}{2(2\theta_1 + 1)} + \sum_{k \neq j}^{2N} \left( \frac{\partial_k}{x_k - x_j} - \frac{\theta_1}{(x_k - x_j)^2} \right) \right] F(x_1, \dots, x_{2N}) = 0, \quad j \in \{1, 2, \dots, 2N\}, \quad (5)$$

where the *one-leg boundary weight*  $\theta_1$  is the conformal weight of the one-leg boundary operator and is given by [2–4] (see appendix A)

$$\theta_1 = \frac{1}{16} \left[ 5 - c \pm \sqrt{(c-1)(c-25)} \right]. \quad (6)$$

Here,  $c$  is the central charge of the CFT that corresponds to the model under consideration. For example,  $c = 0$  corresponds to percolation [1], and  $c = 1/2$  corresponds to the Ising model (i.e., the two-state Potts model) and two-state random cluster model [2]. Also, the sign to be used in (6) depends on the model. For example, we use the  $-$  sign for the Potts model and the  $+$  sign for the random cluster model.

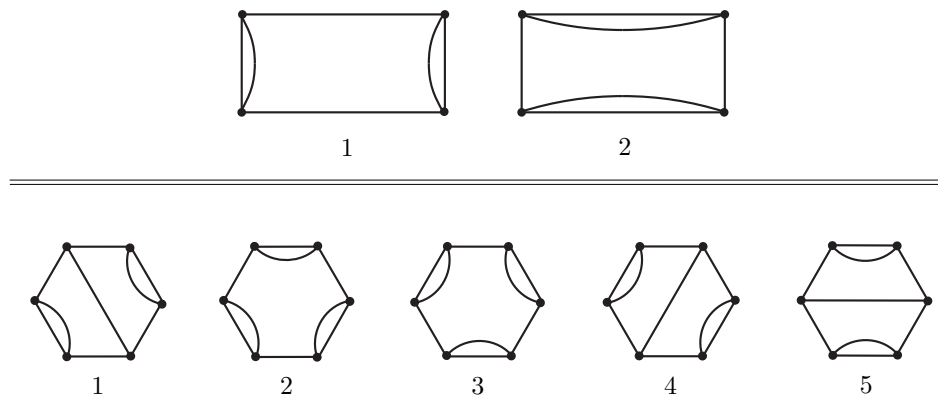


FIG. 3: The number of possible crossing configurations (or equivalently, boundary arc connectivities) in the rectangle (resp. hexagon) equals the second (resp. third) Catalan number  $C_2 = 2$  (resp.  $C_3 = 5$ ).

Aside from the null-state PDEs (5), any CFT correlation function must satisfy three conformal Ward identities, given by [2–4]

$$\sum_{k=1}^{2N} \partial_i F(x_1, \dots, x_{2N}) = 0, \quad \sum_{k=1}^{2N} (x_k \partial_k + \theta_1) F(x_1, \dots, x_{2N}) = 0, \quad \sum_{k=1}^{2N} (x_k^2 \partial_k + 2\theta_1 x_k) F(x_1, \dots, x_{2N}) = 0, \quad (7)$$

to ensure that the correlation function  $F$  is covariant with respect to conformal automorphisms of the upper half-plane, with each point  $x_i$  having conformal weight  $\theta_1$ . This is explained in further detail in the next section of the introduction I below.

When  $N = 1$ , it is easy to show that any solution to the system (5, 7) is of the form  $C(x_2 - x_1)^{-2\theta_1}$  for some arbitrary constant  $C$ . Thus, the rank (i.e., the dimension of the solution space) equals the first Catalan number,  $C_1 = 1$ . When  $N = 2$ , we can use the Ward identities (7) to convert the system of four null-state PDEs (5) into a single hypergeometric differential equation [2]. The general solution of this differential equation completely determines the solution space, so the rank of the system equals the second Catalan number,  $C_2 = 2$ . (After setting  $c = 0$ , an appropriate boundary condition argument gives Cardy’s formula (2).) When  $N = 3$ , a similar but more complicated argument [16] shows that the rank of the system equals the third Catalan number,  $C_3 = 5$ , at least when  $c = 0$ . Beyond this, the need for  $C_N$  linearly independent crossing formulas implies that the rank of the system is at least  $C_N$ , but it does not apparently imply that the rank is  $C_N$  exactly. In spite of this, the Coulomb gas formalism [20, 21] allows us to construct many explicit classical (in the sense of [22]) solutions of the system for arbitrarily large  $N$  [16], a remarkable feat! This article does not work with these solutions, but its sequel [14] works exclusively with them.

In addition to CFT, we can use multiple-SLE $_{\kappa}$  [23, 24], a generalization of SLE $_{\kappa}$  (Schramm-Löwner Evolution) [25–27], to study the continuum limit of a critical lattice model inside a polygon  $\mathcal{P}$  with an FFBC. When we use this approach, we forsake the boundary clusters and study their perimeters instead. These perimeters, called *boundary arcs*, are random, fractal curves that fluctuate inside  $\mathcal{P}$ . Their law is conjectured, (and proven for some models in the case of (ordinary) SLE $_{\kappa}$ , see table I), to be that of multiple-SLE $_{\kappa}$ , a stochastic process that simultaneously grows  $2N$  fractal curves, one from each vertex, inside  $\mathcal{P}$ . These curves explore  $\mathcal{P}$  without crossing themselves or each other until they join to form  $N$  distinct non-crossing boundary arcs that connect the vertices of  $\mathcal{P}$  pairwise (figure 4) in a specified connectivity. An induction argument [13] shows that these  $2N$  curves can join in one of  $C_N$  possible connectivities, called *boundary arc connectivities* [28] (figure 3). Furthermore, we identify each boundary arc connectivity with the particular cluster crossing event whose boundary arcs join in that connectivity.

Multiple-SLE $_{\kappa}$  provides a different, rigorous, formalism for calculating some observables that are also found via CFT, and although these two approaches are fundamentally different, they are closely related. The multiple-SLE $_{\kappa}$  process is driven by two entities [23, 24]:

- I The first is stochastic: a collection of  $2N$  absolutely continuous martingales with zero cross-variation and total quadratic variation  $\kappa t$ , with  $\kappa > 0$  the SLE $_{\kappa}$  *speed* or *parameter* and  $t > 0$  the *evolution time*. The equation [39]

$$c = \frac{(6 - \kappa)(3\kappa - 8)}{2\kappa} \quad (8)$$

relates a CFT of central charge  $c < 1$  (resp.  $c = 1$ ) to a multiple-SLE $_{\kappa}$  with one of two possible speeds, one in the *dilute phase*  $\kappa \in (0, 4]$ , and one in the *dense phase*  $\kappa \in (4, \infty)$  (resp. with one speed  $\kappa = 4$  in the dilute phase)

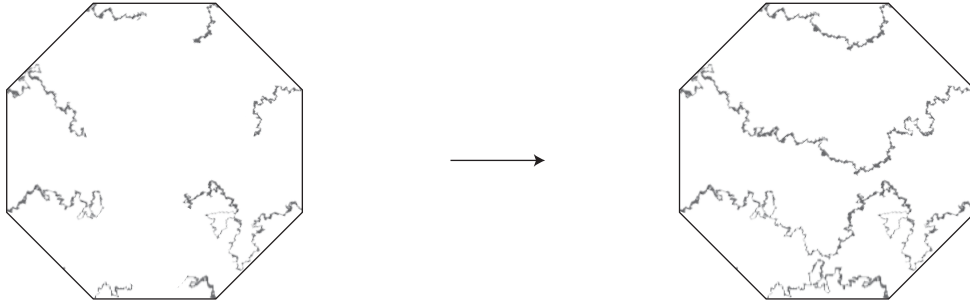


FIG. 4: The evolution of a multiple-SLE $_{\kappa}$  process in an octagon. One multiple-SLE $_{\kappa}$  curve grows from each vertex into the octagon, and these curves eventually join pairwise to form four distinct, non-crossing boundary arcs in one of  $C_4 = 14$  possible boundary arc connectivities.

Random walk or critical lattice model	$\kappa$	$c$	Current status
The loop-erased random walk [29]	2	-2	proven [30]
The self-avoiding random walk [31]	8/3	0	conjectured [32]
$Q = 2$ Potts spin cluster perimeters [18]	3	1/2	proven [33]
$Q = 3$ Potts spin cluster perimeters [18]	10/3	4/5	conjectured [34]
$Q = 4$ Potts spin/FK cluster perimeters [18, 19]	4	1	conjectured [34]
The level line of a Gaussian free field [35]	4	1	proven [35]
The harmonic explorer [35]	4	1	proven [35]
$Q = 3$ Potts FK cluster perimeters [19]	24/5	4/5	conjectured [34]
$Q = 2$ Potts FK cluster perimeters [19]	16/3	1/2	proven [33]
Percolation and smart-kinetic walks [36, 37]	6	0	proven [38]
Uniform spanning trees [30]	8	-2	proven [30]

TABLE I: Models conjectured or proven to have conformally invariant continuum limits and with curves that converge to  $SLE_\kappa$  curves as this continuum limit is approached.

[26] of  $SLE_\kappa$ . Further arguments provided in appendix A show that if we substitute (8) into (6), then we must use the  $-$  (resp.  $+$ ) sign in the dilute (resp. dense) phase, so [39]

$$\theta_1 = \frac{6 - \kappa}{2\kappa}. \quad (9)$$

II The second is deterministic: a function  $F$ , which we call an  $SLE_\kappa$  *partition function*. (This is similar to but slightly different from the actual partition function of the critical system under consideration. See appendix A.) The only condition imposed on  $F$  is that it satisfies the system of null-state PDEs (5) and the three Ward identities (7) (in the classical sense of [22]) with  $\theta_1$  given by (9).

In this article, we focus our attention on the choice of  $SLE_\kappa$  partition function. Because all of the boundary arcs look like  $SLE_\kappa$  curves in the small regardless of our choice, the  $SLE_\kappa$  partition function that we do use can only influence a global property of multiple- $SLE_\kappa$  such as the eventual pairwise connectivity of its curves in the long-time limit. Indeed, two multiple- $SLE_\kappa$  processes whose curves are conditioned to join in different connectivities obey the same stochastic PDEs driven by conditions I–II. Because which  $SLE_\kappa$  partition function to use for condition II is the only unspecified feature of these equations, we expect that this choice influences the connectivity.

This supposition naturally leads us to conjecture the rank of the system (5, 7) previously considered in our CFT approach above. As mentioned, there are  $C_N$  possible connectivities. Thus, there must be at least one  $SLE_\kappa$  partition function per connectivity that will condition the boundary arcs to join in that connectivity almost surely. But furthermore, if our  $SLE_\kappa$  partition function does not influence any of the boundary arcs' other global properties, then there can be at most one  $SLE_\kappa$  partition function  $\Pi_k$ , called the  $k$ th *crossing weight*, that conditions the boundary arcs to join in, say, the  $k$ th connectivity. Assuming that this is true leads us to conclude that the solution space of the system (5, 7) is spanned by  $\{\Pi_1, \dots, \Pi_{C_N}\}$ , and the rank of the system is therefore  $C_N$ . Proving this last statement is one of the principal goals of this article and its sequel [14].

### A. Objectives and organization

So far, we have used the application of the system (5, 7) to critical lattice models and multiple- $SLE_\kappa$  to anticipate some of the properties of its solution space. In this section, we set the stage for our proof of some of these properties by declaring the goals, describing the organization, and establishing some notation conventions for this article and its sequel [14]. Inserting (9) in the system (5, 7) gives the  $2N$  null-state PDEs

$$\left[ \frac{\kappa}{4} \partial_j^2 + \sum_{k \neq j}^{2N} \left( \frac{\partial_k}{x_k - x_j} - \frac{(6 - \kappa)/2\kappa}{(x_k - x_j)^2} \right) \right] F(\mathbf{x}) = 0, \quad j \in \{1, 2, \dots, 2N\}, \quad (10)$$

with  $\mathbf{x} := (x_1, \dots, x_{2N})$  and  $\kappa > 0$ , and the three Ward identities

$$\sum_{k=1}^{2N} \partial_k F(\mathbf{x}) = 0, \quad \sum_{k=1}^{2N} \left[ x_k \partial_k + \frac{6 - \kappa}{2\kappa} \right] F(\mathbf{x}) = 0, \quad \sum_{k=1}^{2N} \left[ x_k^2 \partial_k + \frac{(6 - \kappa)x_k}{\kappa} \right] F(\mathbf{x}) = 0. \quad (11)$$

We call the  $j$ th null-state PDE among (5) *the null-state PDE centered on  $x_j$* . Before declaring what we intend to prove about this system of PDEs, we observe some important facts about it.

- The subsystem of  $2N$  null-state PDEs (10) is undefined on the locus of *diagonal points* in  $\mathbb{R}^{2N}$ , or points with at least two of its coordinates equal. We let  $\Omega$  be the complement of the locus of diagonal points in  $\mathbb{R}^{2N}$ . Then the diagonal points make up the boundary  $\partial\Omega$ , and all together, they divide  $\Omega$  into connected components, each of the form

$$\Omega_\sigma := \{\mathbf{x} \in \Omega \mid x_{\sigma(1)} < x_{\sigma(2)} < \dots < x_{\sigma(2N-1)} < x_{\sigma(2N)}\} \quad (12)$$

for some permutation  $\sigma \in S_{2N}$ . By symmetry, it suffices to restrict the domain of our solutions to the component  $\Omega_0 := \Omega_{\sigma_0}$  corresponding to the identity permutation  $\sigma_0$ . That is, we take  $x_i < x_j$  whenever  $i < j$  without loss of generality. In this article, we refer to  $\mathbf{x} := (x_1, \dots, x_{2N}) \in \Omega_\sigma$  as a point in a component of  $\Omega$  and  $x_i$  as the  $i$ th coordinate of that point, but in the sequel [14], we will refer to  $x_i$  as a point.

- The subsystem (10) is elliptic, so all of its solutions exhibit strong regularity. Indeed, after summing over all  $2N$  null-state PDEs, we find that any solution satisfies a linear homogeneous strictly elliptic PDE whose coefficients are analytic in any connected component of  $\Omega$ . (In fact, the principal part of this PDE is simply the Laplacian.) It follows from the theorem of Hans Lewy [40] that all of its solutions are (real) analytic in any connected component of  $\Omega$ . We will use this fact to exchange the order of integration and differentiation in many of the integral equations that we will encounter later.
- We can explicitly solve the Ward identities (11) via the method of characteristics. It follows that any function  $F : \Omega_0 \rightarrow \mathbb{R}$  that satisfies these identities must have the form [2]

$$F(\mathbf{x}) = G(\eta_1, \dots, \eta_{2N-3}) \prod_{j=1}^{2N} |x_j - x_{\tau(j)}|^{(\kappa-6)/2\kappa}, \quad (13)$$

where  $\{\eta_1, \dots, \eta_{2N-3}\}$  is any set of  $2N - 3$  independent cross-ratios that can be formed from  $x_1, \dots, x_{2N}$ ,  $G(\eta_1, \dots, \eta_{2N-3})$  is a (real) analytic function of  $\mathbf{x} \in \Omega_0$ , and  $\tau$  is any pairing (i.e., a permutation  $\tau \in S_{2N}$  with  $\tau = \tau^{-1}$ ) of the indices  $1, \dots, 2N$ .

- We suppose that  $f$  is a Möbius transformation sending the upper half-plane onto itself, and we define  $x'_i := f(x_i)$  and  $\mathbf{x}' := (x'_1, \dots, x'_{2N})$ . Then the mapping  $T : \Omega_0 \rightarrow \Omega$  defined by  $T(\mathbf{x}) = \mathbf{x}'$  sends  $\Omega_0$  onto a possibly different connected component  $T(\Omega_0)$  of  $\Omega$ . Because the cross-ratios  $\eta_1, \dots, \eta_{2N-3}$  are invariant under  $f$ , the right side of (13) evaluated at any  $\mathbf{x}' \in T(\Omega_0)$  is well-defined. If we enumerate the permutations in  $S_{2N}$  so  $\Omega_0, \Omega_{\sigma_1}, \dots, \Omega_{\sigma_M}$  are all of the components of  $\Omega$  that can be reached from  $\Omega_0$  by such a transformation  $T$ , then we can use (13) to extend  $F$  to the function

$$\hat{F} : \bigcup_{j=0}^M \Omega_{\sigma_j} \rightarrow \mathbb{R}, \quad \hat{F}(\mathbf{x}) := G(\eta_1, \dots, \eta_{2N-3}) \prod_{j=1}^{2N} |x_j - x_{\tau(j)}|^{(\kappa-6)/2\kappa}. \quad (14)$$

It is evident that because  $F$  in (13) satisfies the system of PDEs (10–11),  $\hat{F}$  must satisfy this system too. Now, it is easy to show that (14) transforms covariantly with respect to conformal automorphisms of the upper half-plane, with each of the  $2N$  independent variables having *conformal weight*  $\theta_1$  (9). In other words, the functional equation (where  $\partial f(x) := \partial f(x)/\partial x$ )

$$\hat{F}(\mathbf{x}') = \partial f(x_1)^{-\theta_1} \dots \partial f(x_{2N})^{-\theta_1} \hat{F}(\mathbf{x}), \quad \theta_1 := (6 - \kappa)/2\kappa, \quad (15)$$

holds whenever  $f$  is a Möbius transformation taking the upper half-plane onto itself. Such transformations are compositions of translation by  $a \in \mathbb{R}$ , dilation by  $b > 0$ , and the inversion  $x \mapsto -1/x$  (all of which have positive derivatives). In other words,  $F(\mathbf{x})$  is invariant when we translate the coordinates of  $\mathbf{x}$  by the same amount and is covariant with conformal weight  $\theta_1$  (9) when we dilate these coordinates by the same factor or invert them. These three properties are induced by the first, second, and third Ward identities (11) (counting from the left) respectively.

- We can compute the space of solutions for the system of PDEs (10–11) with domain  $\Omega_0$  when  $N = 1, 2$  [24].
  - In the  $N = 1$  case, we use the first Ward identity of (11) (counting from the left) to reduce either PDE in (10) to an second order Euler differential equation in the one variable  $x_2 - x_1$ . The Euler equation has two characteristic powers (given in (29) below), and the second Ward identity of (11) permits only the power  $1 - 6/\kappa$ . Thus, the solution space is

$$\mathcal{S}_1 = \{F : \Omega_0 \rightarrow \mathbb{R} \mid F(x_1, x_2) = C(x_2 - x_1)^{1-6/\kappa} \text{ for some } C \in \mathbb{R}\}. \quad (16)$$

It is easy to show that the elements of  $\mathcal{S}_1$  satisfy the third Ward identity of (11).



– In the  $N = 2$  case, the Ward identities demand that our solutions have the form (13), which we write as

$$F(\mathbf{x}) = (x_4 - x_2)^{1-6/\kappa} (x_3 - x_1)^{1-6/\kappa} G \left( \frac{(x_2 - x_1)(x_4 - x_3)}{(x_3 - x_1)(x_4 - x_2)} \right), \quad (17)$$

with  $G$  an unspecified function. By substituting (17) into any one of the null-state PDEs, we find that  $[\eta(1-\eta)]^{-2/\kappa} G(\eta)$  satisfies a second order hypergeometric differential equation. This restricts  $G$  to a linear combination of two possible functions  $G_1$  and  $G_2$  given by

$$G_1(\eta) = G_2(1-\eta) = \eta^{2/\kappa} (1-\eta)^{1-6/\kappa} {}_2F_1 \left( \frac{4}{\kappa}, 1 - \frac{4}{\kappa}; \frac{8}{\kappa} \middle| \eta \right), \quad (18)$$

with  ${}_2F_1$  the Gauss hypergeometric function. Thus, with  $\eta := (x_2 - x_1)(x_4 - x_3)/(x_3 - x_1)(x_4 - x_2)$ , the solution space is

$$\mathcal{S}_2 = \{F : \Omega_0 \rightarrow \mathbb{R} \mid F(\mathbf{x}) = [(x_4 - x_2)(x_3 - x_1)]^{1-6/\kappa} [C_1 G_1(\eta) + C_2 G_2(\eta)] \text{ for some } C_1, C_2 \in \mathbb{R}\}. \quad (19)$$

Now we define the solution space for the system of PDEs (10–11) that we wish to rigorously characterize in this article:

**Definition 1.** Let  $\mathcal{S}_N$  denote the vector space of all functions  $F : \Omega_0 \rightarrow \mathbb{R}$

- that satisfy the system of PDEs (10–11) (in the classical sense of [22]), and
- such that for each  $F \in \mathcal{S}_N$ , there exists positive constants  $C$  and  $p$  such that

$$|F(\mathbf{x})| \leq C \prod_{i < j}^{2N} |x_j - x_i|^{\mu_{ij}(p)} \quad \text{with} \quad \mu_{ij}(p) := \begin{cases} -p & |x_i - x_j| < 1 \\ +p & |x_i - x_j| \geq 1 \end{cases}, \quad \text{for all } \mathbf{x} \in \Omega_0. \quad (20)$$

One can explicitly construct many putative elements of  $\mathcal{S}_N$  by using the Coulomb gas formalism first proposed by V.S. Dotsenko and V.A. Fateev [20, 21]. This method is non-rigorous, but a proof that these “candidate solutions” indeed satisfy the system of PDEs (10–11) was given later by J. Dubédat [16]. We call these solutions *Coulomb gas solutions*.

The goal of this article and its sequel [14] is to completely determine the space  $\mathcal{S}_N$  for all  $\kappa \in (0, 8)$ . By “determine,” we mean

1. Prove that  $\mathcal{S}_N$  is indeed spanned by Coulomb gas solutions.
2. Prove that  $\dim \mathcal{S}_N = C_N$ .
3. Argue that  $\mathcal{S}_N$  has a basis  $\mathcal{B}_N := \{\Pi_1, \dots, \Pi_{C_N}\}$  consisting of  $C_N$  crossing weights and calculate that basis.

(For all  $\kappa \geq 8$ , the multiple-SLE $_{\kappa}$  curves are space-filling almost surely. Although we suspect that the findings of this article and its sequels [14, 41] are true for all  $\kappa \geq 8$ , our proofs do not carry over to this range.) Proving items 1 and 2 determines the size and content of  $\mathcal{S}_N$ . In this article, we prove the upper bound  $\dim \mathcal{S}_N \leq C_N$ , assuming conjecture 14 (stated below). In the sequel [14], we will prove that  $\dim \mathcal{S}_N = C_N$  and  $\mathcal{S}_N$  is spanned by Coulomb gas solutions, again assuming conjecture 14. In order to prove items 1 and 2, we construct a basis  $\mathcal{B}_N^*$  for the dual space  $\mathcal{S}_N^*$  of linear functionals acting on  $\mathcal{S}_N$ . A heuristic, though non-rigorous, argument shows that the basis  $\mathcal{B}_N$  for  $\mathcal{S}_N$  that is dual to  $\mathcal{B}_N^*$  is comprised of all of the crossing weights. This realization gives a direct method for their computation, as promised in item 3.

We also plan to publish two other articles with results that follow from the analysis of this article and the sequel [14]. One article [15] will use the crossing weights to non-rigorously calculate new crossing formulas for various critical lattice models. The second article [41] will rigorously investigate degenerate behavior of  $\mathcal{S}_N$  that occurs for certain “exceptional” multiple-SLE $_{\kappa}$  speeds corresponding to CFT minimal models. (Exceptional speeds arise naturally in the analysis of [14].)

The appendix consists of three parts. In appendix A, we survey some of the CFT methodologies used to study critical lattice models. This formalism (non-rigorously) anticipates many of our results, so we often interpret our findings in context with CFT throughout this article. The reader who is not familiar with this approach but wishes to understand our asides to it may consult this appendix. In appendix B, we suggest a possible but incomplete proof of conjecture 14. In appendix C, we discuss the power-law bound (20) that defines the solution space  $\mathcal{S}_N$ . Because correlation functions for critical lattice models typically exhibit power-law behavior, this bound is natural in our application. However, we suspect that it is unnecessary and  $\mathcal{S}_N$  is in fact the entire solution space of the system of PDEs (10–11), and we motivate this belief in this last appendix.

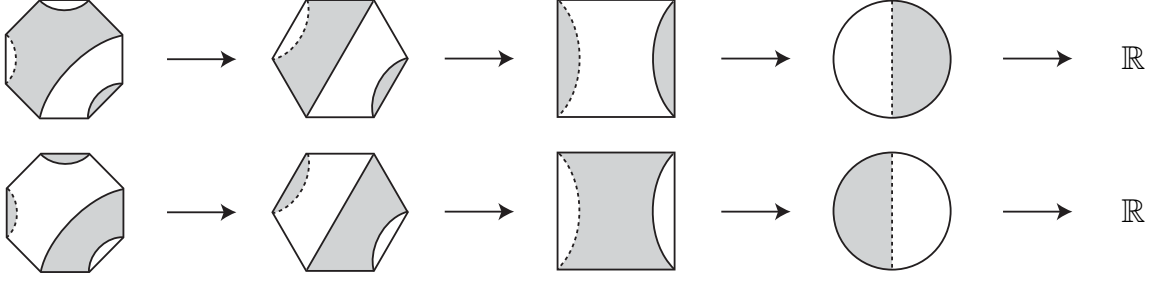


FIG. 5: An illustration of the mapping that sends an octagon crossing weight to a nonzero real number. In each step, the endpoints of the dashed boundary arc of the  $2N$ -sided polygon  $\mathcal{P}$  are brought together to give the  $(2N - 2)$ -sided polygon  $\mathcal{P}'$  immediately to the right of  $\mathcal{P}$ .

### B. A survey of our approach

In this section, we motivate our method to complete items 1–3 (stated in the previous section), momentarily restricting our attention to percolation ( $\kappa = 6$ ) for simplicity.

To begin, we choose one of the  $C_N$  crossing configurations in a  $2N$ -sided polygon  $\mathcal{P}$  with vertices  $w_1, \dots, w_{2N} \in \mathbb{C}$ . Topological considerations show that there are at least two sides of  $\mathcal{P}$  whose two adjacent vertices are endpoints of a common boundary arc (i.e., multiple-SLE $_{\kappa}$  curve), and we let  $[w_i, w_{i+1}]$  be such a side.

Next, we investigate what happens when the vertices  $w_i$  and  $w_{i+1}$  approach each other. What we find depends on the boundary condition for  $[w_i, w_{i+1}]$ . If this side is wired, then the adjacent free sides  $[w_{i-1}, w_i]$  and  $[w_{i+1}, w_{i+2}]$  fuse into one contiguous free side  $[w_{i-1}, w_{i+2}]$  of a  $(2N - 2)$ -sided polygon  $\mathcal{P}'$ , and the isolated boundary cluster previously anchored to  $[w_i, w_{i+1}]$  contracts away. Or if  $[w_i, w_{i+1}]$  is free, then the adjacent wired sides fuse into one contiguous wired side of  $\mathcal{P}'$ , and the boundary clusters previously anchored to these sides fuse into one boundary cluster anchored to this new wired side. In either situation, the original crossing configuration for the  $2N$ -sided polygon goes to a crossing configuration for a  $(2N - 2)$ -sided polygon, and the crossing weight for the former configuration goes to the crossing weight for the latter configuration (figure 5). (In percolation, “crossing weight” and “crossing formula” are synonymous.) If we repeat this process  $N - 1$  more times, then we end with a zero-sided polygon, or disk, whose boundary is either all wired or all free. The disk trivially exhibits just one “crossing” configuration, so this cumulative process sends the original crossing weight to one.

Now, we can envision bringing together in a different order the same pairs of vertices that approach each other. But because all of these variations send the same crossing weight to one, we anticipate that all of them are different realizations of the same map.

Next, we investigate what happens when two adjacent vertices  $w_i$  and  $w_{i+1}$  that are not endpoints of a common boundary arc approach each other. Using the same crossing configuration as before, we observe one of two outcomes. If the side  $[w_i, w_{i+1}]$  is wired, then the adjacent free sides  $[w_{i-1}, w_i]$  and  $[w_{i+1}, w_{i+2}]$  of  $\mathcal{P}$  do not fuse into one contiguous free segment. Instead, they are separated by an infinitesimal wired segment centered on the point  $w_i = w_{i+1}$  within the side  $[w_{i-1}, w_{i+2}]$  of  $\mathcal{P}'$ , and the boundary cluster previously anchored to  $[w_i, w_{i+1}]$  now anchors to this infinitesimal segment. Or if  $[w_i, w_{i+1}]$  is free, then the adjacent wired sides do not fuse into one contiguous wired segment. Instead, they are separated by an infinitesimal free segment centered on  $w_i$ , and the boundary clusters that originally anchored to the adjacent wired sides remain separated by this segment. The likelihood of witnessing either of these two configurations in  $\mathcal{P}'$  with respect to the point  $w_i$  on its boundary, is zero. Hence, pulling together two vertices not connected by a common boundary arc sends the crossing weight for the original configuration in  $\mathcal{P}$  to zero. In CFT, this corresponds to the appearance of just the two-leg channel in the OPE of the one-leg corner operators at

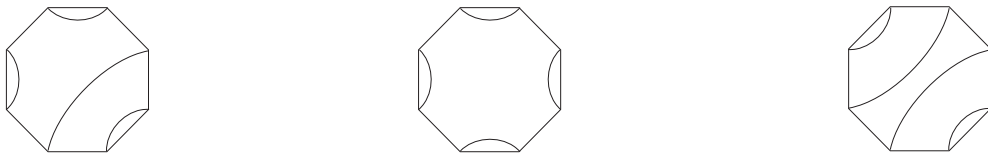


FIG. 6: Three arc connectivity diagrams for the octagon. The other  $C_4 - 3 = 11$  diagrams are found by rotating one of these three.



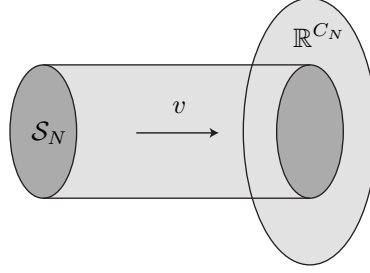


FIG. 7: An illustration of the linear mapping  $v$ . The goal of this article is to show that  $v$  is injective (assuming conjecture 14 below). In the sequel [14], we show that this mapping is an isomorphism (again assuming conjecture 14).

$w_i$  and  $w_{i+1}$  (see appendix A).

All of these mappings that pull pairs of adjacent vertices of  $\mathcal{P}$  together are subject to one constraint: the vertices  $w_{i_{2j-1}}$  and  $w_{i_{2j}}$  of the  $2(N-j+1)$ -sided polygon  $\mathcal{P}_j$  to be brought together at the  $j$ th step of this mapping cannot be separated from each other by any other vertices within the boundary of  $\mathcal{P}_j$ . If we imagine connecting  $w_{i_{2j-1}}$  and  $w_{i_{2j}}$  with an arc in  $\mathcal{P}$  for each  $j \in \{1, \dots, N\}$ , then this condition is satisfied if and only if we can draw these arcs in  $\mathcal{P}$  so they do not intersect. Furthermore, two mappings that bring the same pairs of vertices together in a different order have the same such arc connectivities, of which there are  $C_N$  (4) (figure 6). Now, if the order in which the sides of  $\mathcal{P}$  are collapsed does not affect the image of any of these mappings, then there are effectively only  $C_N$  distinct mappings. Assuming that the order indeed does not matter, we denote the  $k$ th of these mappings by  $[\mathcal{L}_k]$ , and we let  $\mathcal{B}_N^* := \{[\mathcal{L}_1], \dots, [\mathcal{L}_{C_N}]\}$ .

If the arc connectivity for some specified mapping in  $\mathcal{B}_N^*$  matches (resp. does not match) the boundary arc connectivity for some specified crossing weight, then our arguments imply that this mapping sends that crossing weight to one (resp. zero). Hence, we anticipate that  $[\mathcal{L}_k]\Pi_{k'} = \delta_{k,k'}$  for all  $k, k' \in \{1, \dots, C_N\}$  and, assuming that  $\mathcal{S}_N^*$  is finite-dimensional with basis  $\mathcal{B}_N^*$ , the set  $\mathcal{B}_N$  of crossing weights is the basis of  $\mathcal{S}_N$  dual to  $\mathcal{B}_N^*$ . (Because the argument for this dual relation is non-rigorous,  $\Pi_k$  (resp.  $\mathcal{B}_N$ ) will technically stand for the element (resp. basis) of  $\mathcal{S}_N$  dual to  $[\mathcal{L}_k]$  (resp.  $\mathcal{B}_N^*$ ) in the sequel [14], though we will still refer to  $\Pi_k$  as a crossing weight in our discussions.)

If  $\mathcal{B}_N^*$  is a basis for  $\mathcal{S}_N^*$ , then this duality relation interprets  $[\mathcal{L}_k]F$  for every  $F \in \mathcal{S}_N$  as the coefficient of the  $k$ th crossing weight  $\Pi_k$  in the decomposition of  $F$  over  $\mathcal{B}_N$ . Furthermore, the linear mapping  $v : \mathcal{S}_N \rightarrow \mathbb{R}^{C_N}$  with the  $k$ th coordinate of  $v(F)$  equaling  $[\mathcal{L}_k]F$  is injective. That is, although  $v$  destroys the pointwise information contained in each element of  $\mathcal{S}_N$ , it preserves the linear relations between these elements.

Our strategy for proving items 1–3 in section IA is motivated by this reasoning, but our steps are ordered differently because we do not know how to prove that  $\mathcal{B}_N^*$  is a basis for  $\mathcal{S}_N^*$  a priori. So after constructing the elements of  $\mathcal{B}_N^*$  in sections II–III, we prove that the linear mapping  $v$  is injective (assuming conjecture 14) in section IV. Then the dimension theorem bounds the dimension of  $\mathcal{S}_N$  by  $C_N$  (figure 7). Finally, to prove that the dimension of  $\mathcal{S}_N$  is indeed  $C_N$ , we use the Coulomb gas formalism to construct  $C_N$  explicit elements of  $\mathcal{S}_N$  and prove that they are linearly independent (again assuming conjecture 14). This last step will be presented in the sequel [14] and will rely on the machinery constructed in this article. That  $\mathcal{B}_N^*$  is a basis for  $\mathcal{S}_N^*$  will follow as a corollary from these results.

## II. BOUNDARY BEHAVIOR OF SOLUTIONS

Motivated by the observations of section IB, we investigate the behavior of elements of  $\mathcal{S}_N$  near certain points in  $\partial\Omega_0$ . If we conformally map the  $2N$ -sided polygon  $\mathcal{P}$  onto the upper half-plane, with its  $i$ th vertex  $w_i$  sent to the  $i$ th coordinate of  $\mathbf{x} \in \Omega_0$ , then the action of bringing together the vertices  $w_{i+1}$  and  $w_i$  sends  $\mathbf{x}$  to the boundary point  $(x_1, \dots, x_{i-1}, x_i, x_i, x_{i+2}, \dots, x_{2N})$ . This point is in the hyperplane within  $\partial\Omega_0$ , whose points have only the  $i$ th and  $(i+1)$ th coordinates equal. Hence, to implement the mappings described in the previous section for any  $F \in \mathcal{S}_N$ , we must study the limit of  $F(\mathbf{x})$  as  $x_{i+1} \rightarrow x_i$  for any  $i \in \{1, \dots, 2N-1\}$  first.

Interpreting  $F \in \mathcal{S}_N$  as a half-plane correlation function of  $2N$  one-leg boundary operators allows us to anticipate this limit using CFT. (These correlation functions appear, e.g., on the right side of (A6). See appendix A for further details and a review of the CFT nomenclature that we refer to here.) We envisage the  $i$ th coordinate of  $\mathbf{x} \in \Omega_0$  as hosting a one-leg boundary operator  $\psi_1(x_i)$ . If we send  $x_{i+1} \rightarrow x_i$  for some  $i \in \{1, \dots, 2N-1\}$ , then the operators  $\psi_1(x_i)$  and  $\psi_1(x_{i+1})$  fuse into some combination of an identity operator  $\psi_0(x_i)$  (which is actually independent of  $x_i$ ) and a two-leg boundary operator at  $\psi_2(x_i)$ . After inserting their OPE into the  $2N$ -point function  $F$ , we find the

Frobenius series expansion

$$F(x_1, \dots, x_{2N}) = \langle \psi_1(x_1) \dots \psi_1(x_i) \psi_1(x_{i+1}) \dots \psi_1(x_{2N}) \rangle \quad (21)$$

$$= C_{11}^0 (x_{i+1} - x_i)^{-2\theta_1 + \theta_0} \langle \psi_1(x_1) \dots \psi_0(x_i) \psi_1(x_{i+2}) \dots \psi_{2N}(x_{2N}) \rangle + \dots \quad (22)$$

$$+ C_{11}^2 (x_{i+1} - x_i)^{-2\theta_1 + \theta_2} \langle \psi_1(x_1) \dots \psi_2(x_i) \psi_1(x_{i+2}) \dots \psi_{2N}(x_{2N}) \rangle + \dots \quad (23)$$

Here,  $C_{11}^0$  and  $C_{11}^2$  are arbitrary real constants (for our present purposes),  $\theta_s$  is the conformal weight of the  $s$ -leg boundary operator (A14)

$$\theta_s = \frac{s(2s + 4 - \kappa)}{2\kappa}, \quad (24)$$

and the powers in (22–23) are given below in (29). When appropriately normalized, either (22) or (23) is referred to in CFT as a “conformal block,” and (22) and (23) correspond to the identity and two-leg fusion channel respectively.

Motivated by this interpretation, we suppose that  $F \in \mathcal{S}_N$  has a Frobenius series expansion in  $x_{i+1}$  centered on  $x_i$ :

$$F(x_1, \dots, x_{2N}) = (x_{i+1} - x_i)^p F_0(x_1, \dots, x_i, x_{i+2}, \dots, x_{2N}) \quad (25)$$

$$+ (x_{i+1} - x_i)^{p+1} F_1(x_1, \dots, x_i, x_{i+2}, \dots, x_{2N}) \quad (26)$$

$$+ (x_{i+1} - x_i)^{p+2} F_2(x_1, \dots, x_i, x_{i+2}, \dots, x_{2N}) + \dots \quad (27)$$

Using the null-state PDEs (10) centered on  $x_i$  and  $x_{i+1}$  (i.e., with  $j = i$  and  $j = i + 1$ ) and collecting the leading order contributions, we find from either equation that

$$\left[ \frac{\kappa}{4} p(p-1) + p - \frac{6-\kappa}{2\kappa} \right] F_0 = 0. \quad (28)$$

Solving this equation for  $p$ , we find the two powers

$$p = \begin{cases} p_1 = -2\theta_1 + \theta_0 = 1 - 6/\kappa & \text{the identity channel,} \\ p_2 = -2\theta_1 + \theta_2 = 2/\kappa & \text{the two-leg channel,} \end{cases} \quad (29)$$

that appear in (22) and (23) respectively. In this article, we restrict our attention to the range  $\kappa \in (0, 8)$  over which  $p_1 = 1 - 6/\kappa$  is the smaller power. To next order, the null-state PDEs centered on  $x_i$  and  $x_{i+1}$  respectively give

$$-\kappa p \partial_i F_0 + \left[ \frac{\kappa}{4} (p+1)p + (p+1) - \frac{6-\kappa}{2\kappa} \right] F_1 = 0, \quad (30)$$

$$-2\partial_i F_0 + \left[ \frac{\kappa}{4} (p+1)p + (p+1) - \frac{6-\kappa}{2\kappa} \right] F_1 = 0. \quad (31)$$

Taking their difference when  $p = p_1 = 1 - 6/\kappa$  gives  $\partial_i F_0 = 0$ , so  $F_1 = 0$ . In the CFT language, the condition that  $F_1 = 0$  is equivalent to the vanishing of the level-one descendant of the identity operator, and the condition  $\partial_i F_0 = 0$  implies that the identity operator is nonlocal. This latter fact and comparing (25) with (22) suggest that we interpret  $F_0$  as a  $(2N - 2)$ -point function of one-leg boundary operators. If this supposition is true, then  $F_0$  must satisfy the system of PDEs (10–11) in the coordinates  $\{x_j\}_{j \neq i, i+1}$  and with  $N$  replaced by  $N - 1$ . This observation echoes our previous claim that the limit  $x_{i+1} \rightarrow x_i$  (previously  $w_{i+1} \rightarrow w_i$ ) sends a crossing weight for a  $2N$ -sided polygon to that of a  $(2N - 2)$ -sided polygon. If  $p = p_2 = 2/\kappa$  instead, then (30) and (31) are identical, so  $\partial_i F$  is typically not zero. In the CFT language, this implies that the two-leg operator is local. We focus our attention on the  $p = p_1$  case of (29) for now and postpone consideration of the  $p = p_2$  case to appendix B.

These heuristic calculations suggest that for all  $F \in \mathcal{S}_N$ , if we let  $x_{i+1}$  approach  $x_i$  with the values of  $x_i$  and the other coordinates fixed, then  $F(\mathbf{x})$  will either grow or decay with power  $1 - 6/\kappa$  or greater. Lemma 3 below establishes this fact, but before we prove it, we introduce some convenient notation.

**Notation 2.** Let  $\pi_i : \Omega \rightarrow \mathbb{R}^{2N-1}$  and  $\pi_{ij} : \Omega \rightarrow \mathbb{R}^{2N-2}$  be the projection maps removing the  $i$ th coordinate and both the  $i$ th and  $j$ th coordinates respectively from  $\mathbf{x} \in \Omega$ :

$$\begin{aligned} \pi_i(\mathbf{x}) &= (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{2N}), \\ \pi_{ij}(\mathbf{x}) &= (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{j-1}, x_{j+1}, \dots, x_{2N}). \end{aligned}$$

More generally, let  $\pi_{i_1 \dots i_M} : \Omega \rightarrow \mathbb{R}^{2N-M}$  be the projection map removing the coordinates with indices  $i_1, \dots, i_M$  from  $\mathbf{x} \in \Omega$ .

In this article, we will often identify  $\pi_{i+1}(\Omega_0)$  with the subset of the boundary of  $\Omega_0$  whose points have only two coordinates, the  $i$ th and the  $(i+1)$ th, equal. Furthermore, we will sometimes identify  $\pi_{i,i+1}(\Omega_0)$ , explicitly given by

$$\pi_{i,i+1}(\Omega_0) := \{(x_1, \dots, x_{i-1}, x_{i+2}, \dots, x_{2N}) \in \mathbb{R}^{2N-2} \mid x_1 < \dots < x_{i-1} < x_{i+2} < \dots < x_{2N}\}, \quad (32)$$

with the subset of the boundary of  $\Omega_0$  whose points have only three coordinates, the  $(i-1)$ th, the  $i$ th and the  $(i+1)$ th, equal.

**Lemma 3.** *Suppose that  $F \in \mathcal{S}_N$  and  $\kappa \in (0, 8)$ , and for some  $i \in \{1, \dots, 2N-1\}$ , let*

$$\mathbf{x}_\delta := (x_1, \dots, x_i, x_i + \delta, x_{i+2}, \dots, x_{2N}). \quad (33)$$

*Then for all  $j, k \neq i, i+1$  and any compact subset  $\mathcal{K}$  of  $\pi_{i+1}(\Omega_0)$ , the supremums*

$$\sup_{\mathcal{K}} |F(\mathbf{x}_\delta)|, \quad \sup_{\mathcal{K}} |\partial_j F(\mathbf{x}_\delta)|, \quad \sup_{\mathcal{K}} |\partial_j \partial_k F(\mathbf{x}_\delta)| \quad (34)$$

*are  $O(\delta^{1-6/\kappa})$  as  $\delta \downarrow 0$ .*

*Proof.* For each point  $\mathbf{x}_\delta \in \Omega_0$ , we let  $x := x_i$ , we relabel the variables  $\{x_j\}_{j \neq i, i+1}$  as  $\{\xi_1, \xi_2, \dots, \xi_{2N-3}, \xi_{2N-2}\}$  in ascending order, and we let  $\boldsymbol{\xi} := (\xi_1, \dots, \xi_{2N-2})$ . With this new notation, we define

$$F(\boldsymbol{\xi}; x, \delta) := F(\xi_1, \dots, \xi_{i-1}, x, x + \delta, \xi_i, \dots, \xi_{2N-2}). \quad (35)$$

Finally, we choose an arbitrary compact subset  $\mathcal{K} \subset \pi_{i+1}(\Omega_0)$ . Our goal is to prove that the quantities in (34) are  $O(\delta^{1-6/\kappa})$  as  $\delta \downarrow 0$ .

Using the first Ward identity (11), we write the null-state PDE (10) centered on  $x_{i+1}$  as  $\mathcal{L}[F] = \mathcal{M}[F]$ , where

$$\mathcal{L} := \frac{\kappa}{4} \partial_\delta^2 + \frac{\partial_\delta}{\delta} - \frac{(6-\kappa)/2\kappa}{\delta^2}, \quad \mathcal{M} := \sum_j \left( \frac{(6-\kappa)/2\kappa}{(\xi_j - x - \delta)^2} - \frac{(\xi_j - x) \partial_j}{\delta(\xi_j - x - \delta)} \right). \quad (36)$$

Thinking of  $\delta$  as a time variable propagating backwards from an initial condition at some positive  $b$  to zero, we can invert the Euler differential operator  $\mathcal{L}$  with a Green function that satisfies the adjoint problem

$$\mathcal{L}^*[G](\delta, \eta) := \left[ \frac{\kappa}{4} \partial_\eta^2 - \frac{\partial_\eta}{\eta} - \frac{(6-\kappa)/2\kappa - 1}{\eta^2} \right] G(\delta, \eta) = D(\eta - \delta), \quad (37)$$

with  $D$  the Dirac delta function, and with  $\delta > 0$  now a parameter. We choose the initial conditions  $G(\delta, 0) = \partial_\eta G(\delta, 0) = 0$ . Also,  $G$  must be continuous at  $\eta = \delta$ , and  $\partial_\eta G$  must have a jump discontinuity of  $4/\kappa$  at  $\eta = \delta$  in order to satisfy the adjoint equation (37). The unique solution to this initial value problem is

$$G(\delta, \eta) = \frac{4\eta}{8-\kappa} \Theta(\eta - \delta) \left[ \left( \frac{\delta}{\eta} \right)^{1-6/\kappa} - \left( \frac{\delta}{\eta} \right)^{2/\kappa} \right], \quad 0 < \delta, \eta < b. \quad (38)$$

The Heaviside function  $\Theta$  enforces causality (i.e.,  $G(\delta, \eta) = 0$  when  $\eta \leq \delta$ ). Using the usual Green identity [42] we find that for all  $b < 1$  sufficiently small (so the denominators of  $\mathcal{M}[F](\boldsymbol{\xi}; x, \eta)$  are nonzero for all  $(\boldsymbol{\xi}, x) \in \mathcal{K}$  and all  $\eta \in (0, b)$ ),  $F$  satisfies the integral equation

$$F(\boldsymbol{\xi}; x, \delta) = \int_\delta^b G(\delta, \eta) \mathcal{M}[F](\boldsymbol{\xi}; x, \eta) d\eta - \frac{\kappa}{4} [G(\delta, b) \partial_\delta F(\boldsymbol{\xi}; x, b) - \partial_\eta G(\delta, b) F(\boldsymbol{\xi}; x, b)] - \frac{1}{b} G(\delta, b) F(\boldsymbol{\xi}; x, b) \quad (39)$$

for  $0 < \delta < b$ . All of the terms on the right side except the integral are manifestly  $O(\delta^{1-6/\kappa})$  as  $\delta \downarrow 0$ , so we only need to bound the integral. After estimating the coefficients in the integrand and estimating  $G$  for  $\kappa < 8$ , we find

$$|F(\boldsymbol{\xi}; x, \delta)| \leq \frac{4}{8-\kappa} \int_\delta^b \left( \frac{\delta}{\eta} \right)^{1-6/\kappa} |\eta \mathcal{M}[F](\boldsymbol{\xi}; x, \eta)| d\eta + O(\delta^{1-6/\kappa}), \quad 0 < \delta < b. \quad (40)$$

It is natural to define

$$H(\boldsymbol{\xi}; x, \delta) := \delta^{6/\kappa-1} F(\boldsymbol{\xi}; x, \delta) \quad (41)$$

so that proving the lemma amounts to showing that the supremums of  $|H(\boldsymbol{\xi}; x, \delta)|$ ,  $|\partial_j H(\boldsymbol{\xi}; x, \delta)|$ , and  $|\partial_j \partial_k H(\boldsymbol{\xi}; x, \delta)|$  over  $\mathcal{K}$  are bounded over  $0 < \delta < b$ . Now according to (40), there are positive functions  $C_0$  and  $C_1$  such that

$$|H(\boldsymbol{\xi}; x, \delta)| \leq C_1(\boldsymbol{\xi}, x, b) + C_0(\boldsymbol{\xi}, x, b) \int_{\delta}^b |\eta \mathcal{M}[H](\boldsymbol{\xi}; x, \eta)| d\eta, \quad 0 < \delta < b. \quad (42)$$

Next, we bound terms in the integrand in (42) that contain derivatives of  $H$ . For  $j \neq i, i+1$ , the null-state PDE centered on  $x_j$  becomes (now for  $j \in \{1, \dots, 2N-2\}$ )

$$\left[ \frac{\kappa}{4} \partial_j^2 + \sum_{k \neq j} \left( \frac{\partial_k}{\xi_k - \xi_j} - \frac{(6-\kappa)/2\kappa}{(\xi_k - \xi_j)^2} \right) + \frac{\partial_x}{x - \xi_j} - \frac{\delta \partial_{\delta}}{(x - \xi_j)(x + \delta - \xi_j)} - \frac{(6-\kappa)/2\kappa}{(x - \xi_j)^2} - \frac{(6-\kappa)/2\kappa}{(x + \delta - \xi_j)^2} + \frac{6/\kappa - 1}{(x - \xi_j)(x + \delta - \xi_j)} \right] H(\boldsymbol{\xi}; x, \delta) = 0, \quad (43)$$

while that centered on  $x_i$  becomes

$$\left[ \frac{\kappa}{4} (\partial_x - \partial_{\delta})^2 + \frac{\partial_{\delta}}{\delta} + \frac{(6-\kappa)(\partial_x - \partial_{\delta})}{2\delta} + \sum_k \left( \frac{\partial_k}{\xi_k - x} - \frac{(6-\kappa)/2\kappa}{(\xi_k - x)^2} \right) \right] H(\boldsymbol{\xi}; x, \delta) = 0, \quad (44)$$

and that centered on  $x_{i+1}$  becomes

$$\left[ \frac{\kappa}{4} \partial_{\delta}^2 - \frac{(\partial_x - \partial_{\delta})}{\delta} - \frac{(6-\kappa)\partial_{\delta}}{2\delta} + \sum_k \left( \frac{\partial_k}{\xi_k - x - \delta} - \frac{(6-\kappa)/2\kappa}{(\xi_k - x - \delta)^2} \right) \right] H(\boldsymbol{\xi}; x, \delta) = 0. \quad (45)$$

Also, the three Ward identities (11) become

$$\left[ \sum_k \partial_k + \partial_x \right] H(\boldsymbol{\xi}; x, \delta) = 0, \quad (46)$$

$$\left[ \sum_k (\xi_k \partial_k + (6-\kappa)/2\kappa) + x \partial_x + \delta \partial_{\delta} \right] H(\boldsymbol{\xi}; x, \delta) = 0, \quad (47)$$

$$\left[ \sum_k (\xi_k^2 \partial_k + (6-\kappa)\xi_k/\kappa) + x^2 \partial_x + (2x + \delta)\delta \partial_{\delta} \right] H(\boldsymbol{\xi}; x, \delta) = 0. \quad (48)$$

Summing (43) over  $j \in \{1, \dots, 2N-2\}$  and using (46-47) to isolate  $\partial_x H$  and  $\delta \partial_{\delta} H$  in terms of  $H$  and its other derivatives, we find that  $H$  obeys a strictly elliptic linear PDE in the coordinates of  $\boldsymbol{\xi}$  and with  $x$  and  $\delta$  as parameters:

$$\begin{aligned} \sum_j \left[ \frac{\kappa}{4} \partial_j^2 + \sum_{k \neq j} \left( \frac{\partial_k}{\xi_k - \xi_j} - \frac{(6-\kappa)/2\kappa}{(\xi_k - \xi_j)^2} \right) - \sum_k \frac{\partial_k}{x - \xi_j} - \sum_k \frac{(x - \xi_k) \partial_k}{(x - \xi_j)(x + \delta - \xi_j)} \right. \\ \left. + \frac{(N-1)(6/\kappa - 1)}{(x - \xi_j)(x + \delta - \xi_j)} - \frac{(6-\kappa)/2\kappa}{(x - \xi_j)^2} - \frac{(6-\kappa)/2\kappa}{(x + \delta - \xi_j)^2} + \frac{6/\kappa - 1}{(x - \xi_j)(x + \delta - \xi_j)} \right] H(\boldsymbol{\xi}; x, \delta) = 0. \end{aligned} \quad (49)$$

Now we let  $\mathcal{K}_x = \pi_{i,i+1}(\{\mathbf{x} \in \Omega_0 \mid \pi_{i+1}(\mathbf{x}) \in \mathcal{K} \text{ and } x_i = x\})$ , and we let  $\mathcal{U}_1$  and  $\mathcal{U}_0$  be open sets in  $\pi_{i,i+1}(\Omega_0)$  with  $\mathcal{K}_x \subset \subset \mathcal{U}_1 \subset \subset \mathcal{U}_0 \subset \subset \pi_{i,i+1}(\Omega_0)$ . The Schauder interior estimate [22] says that for all  $j, k \in \{1, \dots, 2N-2\}$ ,

$$d \sup_{\mathcal{U}_1} |\partial_j H(\boldsymbol{\xi}; x, \delta)| + d \sup_{\mathcal{U}_1} |\partial_j \partial_k H(\boldsymbol{\xi}; x, \delta)| \leq C(x, \delta, \text{diam } \mathcal{U}_0/2) \sup_{\mathcal{U}_0} |H(\boldsymbol{\xi}; x, \delta)|, \quad (50)$$

where  $d = \text{dist}(\partial \mathcal{U}_0, \partial \mathcal{U}_1)$ ,  $\text{diam } \mathcal{U}_0$  is the diameter of  $\mathcal{U}_0$ , and the function  $C$  depends on  $\delta$  through the supremums of the coefficients of (49) over  $\mathcal{U}_0$ . Because these supremums are bounded as  $\delta \downarrow 0$ , it follows that for fixed  $x$ ,  $C$  is bounded over  $0 < \delta < b$ . Altogether, this implies that there is a function  $C'$  such that

$$\sup_{\mathcal{U}_1} |\delta \mathcal{M}[H](\boldsymbol{\xi}; x, \delta)| \leq d^{-1} C'(x, \text{diam } \mathcal{U}_0/2) \sup_{\mathcal{U}_0} |H(\boldsymbol{\xi}; x, \delta)|, \quad 0 < \delta < b. \quad (51)$$

Taking the supremum of (42) over  $\mathcal{U}_1$ , and using (51), we find that there are positive functions  $c_1$  and  $c_2$  such that

$$\sup_{\mathcal{U}_1} |H(\boldsymbol{\xi}; x, \delta)| \leq c_1(x) + c_2(x) \int_{\delta}^b \sup_{\mathcal{U}_0} |H(\boldsymbol{\xi}; x, \eta)| d\eta, \quad 0 < \delta < b. \quad (52)$$

We use this integral equation to prove the lemma. In the new coordinates, the bound (20) that defines the solution space  $\mathcal{S}_N$  becomes  $|H(\boldsymbol{\xi}; x, \delta)| \leq C_3(\boldsymbol{\xi}, x)\delta^{-p}$  for some  $p \in \mathbb{R}$ , which we take to be an integer without loss of generality, and some positive function  $C_3$  whose supremum over  $\mathcal{U}_0$  is bounded by another positive function  $c_3$ . Therefore,

$$\sup_{\mathcal{U}_0} |H(\boldsymbol{\xi}; x, \delta)| \leq c_3(x)\delta^{-p}, \quad 0 < \delta < b. \quad (53)$$

We assume that  $p$  is positive; otherwise the proof is trivial. If we insert (53) into (52) and integrate, then we find that there is a function  $c_4$  such that

$$\sup_{\mathcal{U}_1} |H(\boldsymbol{\xi}; x, \delta)| \leq c_4(x)\delta^{-p+1}, \quad 0 < \delta < b. \quad (54)$$

If  $p = 1$ , then we stop. If  $p > 1$ , then we choose another open set  $\mathcal{U}_2$  with  $\pi_i(\mathcal{K}_x) \subset \subset \mathcal{U}_2 \subset \subset \mathcal{U}_1$ , and we repeat the previous steps to find that

$$\sup_{\mathcal{U}_2} |H(\boldsymbol{\xi}; x, \delta)| \leq c_5(x)\delta^{-p+2}, \quad 0 < \delta < b, \quad (55)$$

for some function  $c_5$ . If  $p = 2$ , then we stop. If  $p > 2$ , then we repeat this process another  $p - 2$  times to eventually find that

$$\sup_{\mathcal{K}_x} |H(\boldsymbol{\xi}; x, \delta)| \leq \sup_{\mathcal{U}_p} |H(\boldsymbol{\xi}; x, \delta)| \leq c_{p+3}(x), \quad 0 < \delta < b, \quad (56)$$

where  $\mathcal{U}_p \supset \supset \mathcal{K}_x$  is an open set in  $\pi_{i,i+1}(\Omega_0)$ . This fact and the Schauder interior estimate (50) together imply that

$$\sup_{\mathcal{K}_x} |\partial_j H(\boldsymbol{\xi}; x, \delta)| + \sup_{\mathcal{K}_x} |\partial_j \partial_k H(\boldsymbol{\xi}; x, \delta)| \leq c_{p+4}(x, \delta), \quad 0 < \delta < b. \quad (57)$$

Because the supremums of the terms in (39) and the coefficients in (49) over the set  $\{x \in \mathbb{R} \mid \mathcal{K}_x \neq \emptyset\}$  are finite, the supremums of  $c_{p+3}(x)$  and  $c_{p+4}(x, \delta)$  over this set are finite as well. Thus, we take the supremum of (56–57) over this set to find

$$\sup_{\mathcal{K}} |H(\boldsymbol{\xi}; x, \delta)| \leq c_{p+5}, \quad \sup_{\mathcal{K}} |\partial_j H(\boldsymbol{\xi}; x, \delta)| + \sup_{\mathcal{K}} |\partial_j \partial_k H(\boldsymbol{\xi}; x, \delta)| \leq c_{p+6}(\delta), \quad 0 < \delta < b, \quad (58)$$

for some positive constant  $c_{p+5}$  and some positive function  $c_{p+6}$ . Finally, because the terms in (39) and the coefficients in (49) are bounded on  $0 < \delta < b$ ,  $c_{p+6}$  is also bounded on  $0 < \delta < b$ . Hence, we have proved the lemma.  $\square$

Having proven lemma 3, we prove next what the analysis that preceded the statement of this lemma suggests: the limit of  $(x_{i+1} - x_i)^{6/\kappa-1} F(\mathbf{x})$  as  $x_{i+1} \rightarrow x_i$  exists and is independent of  $x_i$ . If this limit is not (resp. is) zero, then in CFT parlance, we say that the identity operator appears (resp. does not appear) in the OPE of  $\psi_1(x_{i+1})$  with  $\psi_1(x_i)$ .

**Lemma 4.** *Suppose that  $F \in \mathcal{S}_N$  and  $\kappa \in (0, 8)$ , and let  $\mathbf{x}_\delta$  be defined as in (33). Then for all  $j \neq i, i+1$ , the limits*

$$\lim_{\delta \downarrow 0} \delta^{6/\kappa-1} F(\mathbf{x}_\delta), \quad \lim_{\delta \downarrow 0} \delta^{6/\kappa-1} \partial_j F(\mathbf{x}_\delta), \quad \lim_{\delta \downarrow 0} \delta^{6/\kappa-1} \partial_j^2 F(\mathbf{x}_\delta) \quad (59)$$

*exist and are approached uniformly over every compact subset of  $\pi_{i+1}(\Omega_0)$ . Furthermore, the limit of  $\delta^{6/\kappa-1} F(\mathbf{x}_\delta)$  as  $\delta \downarrow 0$  does not depend on  $x_i$ .*

*Proof.* We let  $H, \boldsymbol{\xi}, x, \delta$  and  $b$  be as defined in the proof of lemma 3. First, we show that  $H(\boldsymbol{\xi}; x, \delta)$  has a limit as  $\delta \downarrow 0$ . Because  $H(\boldsymbol{\xi}; x, \delta)$  is bounded on  $0 < \delta < b$ , it suffices to show that its superior limit and inferior limit as  $\delta \downarrow 0$  are equal. From (39), we have that

$$H(\boldsymbol{\xi}; x, \delta) = H(\boldsymbol{\xi}; x, b) - \frac{\kappa}{4} J(\delta, b) \partial_\delta H(\boldsymbol{\xi}; x, b) + \int_\delta^b J(\delta, \eta) \mathcal{M}[H](\boldsymbol{\xi}; x, \eta) d\eta, \quad 0 < \delta < b, \quad (60)$$

where  $\mathcal{M}[H](\boldsymbol{\xi}; x, \eta)$  is given in (36), and where  $J$  is the modified Green function

$$J(\delta, \eta) = \frac{4\eta}{8 - \kappa} \Theta(\eta - \delta) \left[ 1 - \left( \frac{\delta}{\eta} \right)^{8/\kappa-1} \right], \quad 0 < \delta, \eta < b. \quad (61)$$

The bracketed factor in  $J(\delta, \eta)$  is bounded above by one because  $\kappa < 8$ , and the supremum of  $|\eta \mathcal{M}[H](\xi; x, \eta)|$  over  $0 < \eta < b$  is finite according to lemma 3. Thus, after taking the supremum of the magnitude of both sides of (60) over  $0 < \delta < b$ , we find that

$$\sup_{0 < \delta < b} |H(\xi; x, \delta) - H(\xi; x, b)| \leq \frac{\kappa}{8 - \kappa} \left[ |b \partial_\delta H(\xi; x, b)| + \frac{b}{\kappa} \sup_{0 < \eta < b} |\eta \mathcal{M}[H](\xi; x, \eta)| \right]. \quad (62)$$

Next, we show that  $b \partial_\delta H(\xi; x, b)$  vanishes as  $b \downarrow 0$ . We can differentiate (60) with respect to  $\delta$  to find a similar integral equation governing  $\delta \partial_\delta H(\xi; x, \delta)$ . (Because  $H$  is an analytic function of  $\delta$ , we can differentiate with respect to  $\delta$  under the integral sign). We find

$$\delta \partial_\delta H(\xi; x, \delta) = \left(\frac{\delta}{b}\right)^{8/\kappa-1} b \partial_\delta H(\xi; x, b) - \frac{4}{\kappa} \int_\delta^b \left(\frac{\delta}{\eta}\right)^{8/\kappa-1} \eta \mathcal{M}[H](\xi; x, \eta) d\eta, \quad 0 < \delta < b. \quad (63)$$

Again, the supremum of  $|\eta \mathcal{M}[H](\xi; x, \eta)|$  over  $0 < \eta < b$  is finite according to lemma 3. Hence, (63) implies that

$$|\delta \partial_\delta H(\xi; x, \delta)| \leq \left(\frac{\delta}{b}\right)^{8/\kappa-1} |b \partial_\delta H(\xi; x, b)| + \frac{2}{|\kappa - 4|} \left( \sup_{0 < \eta < b} |\eta \mathcal{M}[H](\xi; x, \eta)| \right) |b^{2-8/\kappa} \delta^{8/\kappa-1} - \delta| \quad (64)$$

$$\rightarrow 0 \quad \text{as } \delta \downarrow 0 \quad (65)$$

because  $\kappa < 8$ . Therefore, (62) implies that

$$\lim_{b \downarrow 0} \sup_{0 < \delta < b} |H(\xi; x, \delta) - H(\xi; x, b)| = 0, \quad (66)$$

the inferior and superior limits of  $H(\xi; x, \delta)$  as  $\delta \downarrow 0$  are thus equal, and  $H(\xi; x, \delta)$  has a limit, which we call  $H(\xi; x, 0)$ , as  $\delta \downarrow 0$ .

By replacing  $\delta$  with zero and then  $b$  with  $\delta$  in (60) and taking a supremum over a compact subset  $\mathcal{K}$  of  $\pi_{i+1}(\Omega_0)$ , we find

$$\sup_{\mathcal{K}} |H(\xi; x, \delta) - H(\xi; x, 0)| \leq \frac{\kappa}{8 - \kappa} \sup_{\mathcal{K}} |\delta \partial_\delta H(\xi; x, \delta)| + \frac{4}{8 - \kappa} \int_0^\delta \sup_{\mathcal{K}} |\eta \mathcal{M}[H](\xi; x, \eta)| d\eta, \quad 0 < \delta < b. \quad (67)$$

Lemma 3 implies that the integrand is bounded over  $0 < \eta < \delta$ , so the integral vanishes as  $\delta \downarrow 0$ . Next, after taking the supremum of (64) over  $\mathcal{K}$  and using lemma 3 again, we see that the supremum of  $|\delta \partial_\delta H(\xi; x, \delta)|$  over  $\mathcal{K}$  vanishes as  $\delta \downarrow 0$ . Hence, the left side of (67) vanishes as  $\delta \downarrow 0$ , proving that the limit  $H(\xi; x, 0)$  is approached uniformly over  $\mathcal{K}$ . Because  $\mathcal{K}$  may be any compact subset of  $\pi_{i+1}(\Omega_0)$ , it follows that  $H$  extends continuously to  $\Omega_0 \cup \pi_{i+1}(\Omega_0)$  when we naturally define  $H(\xi, x) := H(\xi; x, 0)$  for all  $(\xi, x) \in \pi_{i+1}(\Omega_0)$ .

We can recycle these arguments to show that for all  $j \in \{1, \dots, 2N - 2\}$ ,  $\partial_j H(\xi; x, \delta)$  and  $\partial_j^2 H(\xi; x, \delta)$  approach limits as  $\delta \downarrow 0$  uniformly over compact subsets of  $\pi_{i+1}(\Omega_0)$ . By taking the  $j$ th partial derivative of (60) (because  $H(\xi; x, \delta)$  is analytic in  $\Omega_0$ , we can exchange the order of integration and differentiation), we find an equation similar to (60) but with a few changes. First, the supremum of the new integrand over compact subsets of  $\pi_{i+1}(\Omega_0)$  is also bounded on  $0 < \eta < b$ , according to lemma 3. Second,  $\delta \partial_\delta H(\xi; x, \delta)$  in the first term of (60) will be replaced with  $\delta \partial_\delta \partial_j H(\xi; x, \delta)$ . By taking the  $j$ th partial derivative of (63) and following the reasoning that led to (64), we find that the supremum of  $\delta \partial_\delta \partial_j H(\xi; x, \delta)$  over compact subsets of  $\pi_{i+1}(\Omega_0)$  vanishes as  $\delta \downarrow 0$ . Hence, we may reuse all of the reasoning presented above to show that  $\partial_j H(\xi; x, \delta)$  approaches a limit as  $\delta \downarrow 0$  uniformly over compact subsets of  $\pi_{i+1}(\Omega_0)$ . Finally, we can use the null-state PDE centered on  $x_j$  (43) and (46) to isolate  $\partial_j^2 H(\xi; x, \delta)$  in terms of lower-order derivatives  $\partial_k H(\xi; x, \delta)$  and  $H(\xi; x, \delta)$  and thus prove that the  $j$ th second derivative of  $H$  approaches a limit as  $\delta \downarrow 0$  uniformly over compact subsets of  $\pi_{i+1}(\Omega_0)$ . Thus, we have proven that the limits in (59) exist and are approached uniformly over compact subsets of  $\pi_{i+1}(\Omega_0)$ .

Finally, to prove that the limit  $H(\xi; x) := H(\xi; x, 0)$  does not depend on  $x$ , it suffices to show that  $\partial_x H(\xi; x, \delta)$  vanishes as  $\delta \downarrow 0$ . Indeed, we can use the first Ward identity (46) and the uniformness of the limits in (59) to commute the derivative with respect to  $x$  with the limit as  $\delta \downarrow 0$  to find that  $\partial_x H(\xi; x) = 0$ . To prove that  $H(\xi; x, \delta)$  vanishes as  $\delta \downarrow 0$ , we subtract (45) from (44) to find the following PDE:

$$\left[ \frac{\kappa}{4} \partial_x - \frac{\kappa}{2} \partial_\delta + \frac{8 - \kappa}{2\delta} \right] \partial_x H(\xi; x, \delta) = \sum_k \left[ \frac{\delta \partial_k}{(\xi_k - x)(\xi_k - x - \delta)} + \frac{[\delta + 2(x - \xi_k)]\delta(6 - \kappa)/2\kappa}{(\xi_k - x)^2(\xi_k - x - \delta)^2} \right] H(\xi; x, \delta). \quad (68)$$

We choose a positive  $a < \min\{x - \xi_{i-1}, (b - \delta)/2\}$ , and we let  $Z(t) := \partial_x H(\xi; x - t, \delta + 2t)$  for  $t \in [0, a]$ . Upon evaluating (68) at  $(\xi; x, \delta) \mapsto (\xi, x - t, \delta + 2t)$  and multiplying both sides by  $-4(\delta + 2t)^{1-8/\kappa}/\kappa$ , we find



$$\begin{aligned} \frac{d}{dt} [(\delta + 2t)^{1-8/\kappa} Z(t)] = & -\frac{4}{\kappa} (\delta + 2t)^{2-8/\kappa} \sum_k \left[ \frac{\partial_k}{(\xi_k - x + t)(\xi_k - x - \delta - t)} \right. \\ & \left. + \frac{[\delta + 2(x - \xi_k)](6 - \kappa)/2\kappa}{(\xi_k - x + t)^2(\xi_k - x - \delta - t)^2} \right] H(\xi; x - t, \delta + 2t). \end{aligned} \quad (69)$$

Integrating both sides with respect to  $t$  from 0 to  $a$ , we have

$$\begin{aligned} \partial_x H(\xi; x, \delta) = & \left( \frac{\delta}{\delta + 2a} \right)^{8/\kappa-1} \partial_x H(\xi; x - a, \delta + 2a) + \frac{4}{\kappa} \delta^{8/\kappa-1} \\ & \times \int_0^a dt (\delta + 2t)^{2-8/\kappa} \sum_k \left[ \frac{\partial_k}{(\xi_k - x + t)(\xi_k - x - \delta - t)} + \frac{[\delta + 2(x - \xi_k)](6 - \kappa)/2\kappa}{(\xi_k - x + t)^2(\xi_k - x - \delta - t)^2} \right] H(\xi; x - t, \delta + 2t). \end{aligned} \quad (70)$$

Because the sum inside of the integrand is bounded over  $(t, \delta) \in (0, a) \times (0, b)$ , we have that for some positive function  $\Phi(\xi, x, a, b)$ ,

$$|\partial_x H(\xi; x, \delta)| \leq \left( \frac{\delta}{\delta + 2a} \right)^{8/\kappa-1} |\partial_x H(\xi; x - a, \delta + 2a)| + \Phi(\xi, x, a, b) \delta^{8/\kappa-1} \int_0^a dt (\delta + 2t)^{2-8/\kappa}, \quad (71)$$

from which it immediately follows that

$$\partial_x H(\xi; x, \delta) = O(\delta^{8/\kappa-1}) + O(\delta^2) \rightarrow 0 \quad \text{as } \delta \downarrow 0. \quad (72)$$

□

The last equation (72) of the preceding proof may be interpreted in CFT parlance as the two possible fusion channels (i.e., indicial powers for the Frobenius series in (22–23)). Indeed, if we divide (72) by  $\delta^{6/\kappa-1}$  to revert from  $H$  to  $F$  (41), then the first term in (72) corresponds to the two-leg fusion channel because  $-2\theta_1 + \theta_2 = 2/\kappa$ . On the other hand, the power  $-2\theta_1 + \theta_0 - 2 = -1 - 6/\kappa$  of the second term is not  $-2\theta_1 + \theta_0 = 1 - 6/\kappa$ , but it is still indicative of the identity channel for the following reason. Just before stating lemma 3, we showed that if  $F$  admits a Frobenius series expansion with leading power  $1 - 6/\kappa$ , then the function  $F_0$  in the leading term (25) is independent of  $x_i$  and the function  $F_1$  in the following term (26) is zero. Thus, the leading term (27) in  $\partial_i F$  has the indicial power  $-1 - 6/\kappa$ .

The proof of lemma 4 leads to some interesting integral equations that  $H(\xi; x, \delta)$  and  $F(\xi; x, \delta)$  must satisfy. After replacing  $\delta$  with zero and then replacing  $b$  with  $\delta$  in (60), we find

$$H(\xi; x, \delta) = H(\xi; x, 0) + \frac{\kappa\delta}{8 - \kappa} \partial_\delta H(\xi; x, \delta) - \frac{4}{8 - \kappa} \int_0^\delta \eta \mathcal{M}[H](\xi; x, \eta) d\eta. \quad (73)$$

This integral equation is interesting because it integrates over all  $\eta \in (0, \delta)$  instead of over all  $\eta > \delta$  up to some positive cutoff  $b$ . By moving the middle term on the right side of (73) to the left side, rewriting the left side as a derivative, and integrating up to the cutoff  $b$  used in the proofs above, we further find that

$$H(\xi; x, \delta) = \left( \frac{\delta}{b} \right)^{8/\kappa-1} H(\xi; x, b) + \left[ 1 - \left( \frac{\delta}{b} \right)^{8/\kappa-1} \right] H(\xi; x, 0) - \frac{4\delta^{8/\kappa-1}}{\kappa} \int_\delta^b \int_0^\beta \beta^{-8/\kappa} \eta \mathcal{M}[H](\xi; x, \eta) d\eta d\beta \quad (74)$$

for  $0 < \delta < b$ . In terms of  $F(\xi; x, \delta)$ , this is

$$\begin{aligned} F(\xi; x, \delta) = & \left( \frac{\delta}{b} \right)^{2/\kappa} F(\xi; x, b) + \left[ \left( \frac{\delta}{b} \right)^{1-6/\kappa} - \left( \frac{\delta}{b} \right)^{2/\kappa} \right] b^{1-6/\kappa} H(\xi; x, 0) \\ & - \frac{4\delta^{2/\kappa}}{\kappa} \int_\delta^b \int_0^\beta \beta^{-8/\kappa} \eta^{6/\kappa} \mathcal{M}[F](\xi; x, \eta) d\eta d\beta \end{aligned} \quad (75)$$

for  $0 < \delta < b$ . We will use these integral equations in appendix B.

In the discussion preceding lemma 3, the CFT interpretation of  $F \in \mathcal{S}_N$  as a  $2N$ -point function of one-leg boundary operators suggested that the limit of  $(x_{i+1} - x_i)^{6/\kappa-1} F(\mathbf{x})$  as  $x_{i+1} \rightarrow x_i$ , when it is not zero, is a  $(2N - 2)$ -point function of one-leg boundary operators. This motivates the following lemma.

**Lemma 5.** Suppose that  $F \in \mathcal{S}_N$  and  $\kappa \in (0, 8)$ . Then for all  $i \in \{1, \dots, 2N-1\}$ , the function  $F_0 : \pi_{i,i+1}(\Omega_0) \rightarrow \mathbb{R}$  defined by

$$F_0(\pi_{i,i+1}(\mathbf{x})) := \lim_{x_i \rightarrow x_{i-1}} \lim_{x_{i+1} \rightarrow x_i} (x_{i+1} - x_i)^{6/\kappa-1} F(\mathbf{x}) \quad (76)$$

(the second limit being trivial according to lemma 4), is an element of  $\mathcal{S}_{N-1}$ , and the function  $F_0 : \pi_{1,2N}(\Omega_0) \rightarrow \mathbb{R}$  defined by

$$F_0(\pi_{1,2N}(\mathbf{x})) := \lim_{t \rightarrow \infty} (2t)^{6/\kappa-1} F(-t, x_2, \dots, x_{2N-1}, t) \quad (77)$$

is also an element of  $\mathcal{S}_{N-1}$ .

*Proof.* We let  $H, \xi, x, \delta$ , and  $H(\xi; x, 0)$  be defined as in the proof of lemma 3. To begin, we prove that the first limit (76) is in  $\mathcal{S}_{N-1}$ . By sending  $\delta \downarrow 0$  in (43) and (46–48) and using the limits (65, 72), we find equations almost identical to the  $(2N-2)$  null-state PDEs in the coordinates of  $\xi$ ,

$$\left[ \frac{\kappa}{4} \lim_{\delta \downarrow 0} \partial_j^2 + \sum_{k \neq j} \left( \frac{1}{\xi_k - \xi_j} \lim_{\delta \downarrow 0} \partial_k - \frac{(6-\kappa)/2\kappa}{(\xi_k - \xi_j)^2} \lim_{\delta \downarrow 0} \right) \right] H(\xi; x, \delta) = 0, \quad (78)$$

and the three Ward identities also in the coordinates of  $\xi$ :

$$\sum_k \lim_{\delta \downarrow 0} \partial_k H(\xi; x, \delta) = 0, \quad \sum_k \left[ \xi_k \lim_{\delta \downarrow 0} \partial_k + \frac{6-\kappa}{2\kappa} \lim_{\delta \downarrow 0} \right] H(\xi; x, \delta) = 0, \quad \sum_k \left[ \xi_k^2 \lim_{\delta \downarrow 0} \partial_k + \frac{6-\kappa}{\kappa} \xi_k \lim_{\delta \downarrow 0} \right] H(\xi; x, \delta) = 0. \quad (79)$$

According to lemma 4,  $H(\xi; x, \delta)$  and each of its first and second derivatives approach their limits as  $\delta \downarrow 0$  uniformly over compact subsets of  $\pi_{i+1}(\Omega_0)$ . Hence, we may commute the taking of each limit with each differentiation in (78–79) to find that the limit  $H(\xi; x, 0)$  satisfies the  $(2N-2)$  null-state PDEs (10) and the three Ward identities (11) in the coordinates of  $\xi$ . It is also evident that this limit satisfies the bound (20) in the coordinates of  $\xi$ . Thus, it is an element of  $\mathcal{S}_{N-1}$ .

Finally, we can prove (77) by using (15), which we now write as

$$F(\mathbf{x}) = |\partial f(x_1)|^{(6-\kappa)/2\kappa} \dots |\partial f(x_{2N})|^{(6-\kappa)/2\kappa} \hat{F}(\mathbf{x}') \quad (80)$$

with  $\hat{F}$  defined in (14),  $x' := f(x)$ ,  $\mathbf{x}' := (x'_1, \dots, x'_{2N})$ , and for our present purposes, with  $f$  the Möbius transformation

$$f(x) = \frac{(x_{2N-1} + 1 - x_{2N-2})(x - x_{2N})}{(x_{2N} - x_{2N-2})(x - x_{2N-1} - 1)} \quad (81)$$

cyclically permuting the coordinates of  $\mathbf{x}$  rightward along the real axis so that  $x'_{2N} = 0 < x'_1 < x'_2 < \dots < x'_{2N-2} = 1 < x'_{2N-1}$ . (We note that  $\mathbf{x}'$  is not in  $\Omega_0$  because  $x'_{2N} < x'_1$ .) From (80), we find

$$(2t)^{6/\kappa-1} F(-t, x_2, \dots, x_{2N-1}, t) \underset{t \rightarrow \infty}{\sim} |\partial f(x_2)|^{(6-\kappa)/2\kappa} \dots |\partial f(x_{2N-1})|^{(6-\kappa)/2\kappa} (x'_1 - x'_{2N})^{6/\kappa-1} \hat{F}(x'_1, \dots, x'_{2N}). \quad (82)$$

In the primed coordinates,  $x'_1 \rightarrow x'_{2N}$  as  $t \rightarrow \infty$ . Because  $\hat{F}$  satisfies the system of PDEs (10–11) and obeys the bound (20) in the primed coordinates, we can invoke the result of the previous paragraph to conclude that

$$\begin{aligned} & \lim_{t \rightarrow \infty} (2t)^{6/\kappa-1} F(-t, x_2, \dots, x_{2N-1}, t) \\ &= |\partial f_0(x_2)|^{(6-\kappa)/2\kappa} \dots |\partial f_0(x_{2N-1})|^{(6-\kappa)/2\kappa} F_0(X'_2, X'_3, \dots, X'_{2N-1}), \quad X'_j := \lim_{t \rightarrow \infty} x'_j, \end{aligned} \quad (83)$$

for some  $F_0 \in \mathcal{S}_{N-1}$ . Here,  $f_0$  is the limit of (81) evaluated at  $x_{2N} = t$  as  $t \rightarrow \infty$ , so  $f_0(x_j) = X'_j$  for all  $j \neq 1, 2N$ . Transformation law (80) adapted to  $F_0$  becomes the functional equation

$$|\partial f_0(x_2)|^{(6-\kappa)/2\kappa} \dots |\partial f_0(x_{2N-1})|^{(6-\kappa)/2\kappa} F_0(X'_2, \dots, X'_{2N-1}) = F_0(x_2, \dots, x_{2N-1}). \quad (84)$$

(We have removed the hat that would appear above  $F_0$  in (83) and on the left side of (84) because  $X'_2 < X'_3 < \dots < X'_{2N-1}$ .) Combining (83) and (84), we find that (77) is an element of  $\mathcal{S}_{N-1}$ .  $\square$

For notational convenience, we will omit explicit reference to the trivial limit  $x_i \rightarrow x_{i-1}$  in (76) from now on.

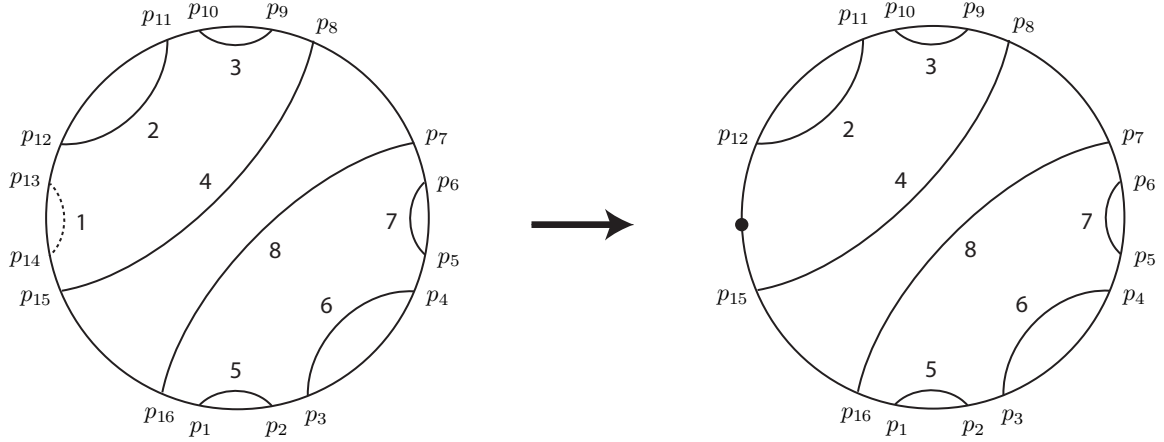


FIG. 8: An interior arc connectivity diagram for some  $\mathcal{L} \in \mathcal{S}_8^*$ . The endpoints of the  $j$ th arc are brought together for the  $j$ th limit of  $\mathcal{L}$ , and the figure illustrates the first limit.

### III. CONSTRUCTION OF THE DUAL SPACE $\mathcal{S}_N^*$

Inspired by the special property conveyed in lemma 5, we construct elements of the dual space  $\mathcal{S}_N^*$  as follows. Starting with any element of  $\mathcal{S}_N$ , we take the limit (76) or (77) to find an element of  $\mathcal{S}_{N-1}$ . Then we repeat this process  $N - 1$  more times until we ultimately arrive with an element of  $\mathcal{S}_0 = \mathbb{R}$ . This sequence of limits is a linear mapping  $\mathcal{L}$  sending  $\mathcal{S}_N$  into the real numbers and is thus an element of  $\mathcal{S}_N^*$ . (In the sequel [14], we will show that the set of all such mappings is a basis for  $\mathcal{S}_N^*$  (assuming conjecture 14 below), as the title of this section suggests.)

The  $N$  limits of  $\mathcal{L}$  must be carefully ordered in order for the action of this map on elements of  $\mathcal{S}_N$  to be well-defined. Intuitively, this restriction is best understood by drawing a disk and marking on its boundary  $2N$  points  $P = \{p_1, \dots, p_{2N}\}$  in counterclockwise order and with  $p_i$  corresponding to the  $i$ th coordinate of  $\mathbf{x} \in \Omega_0$  (figure 8). Then the first limit of  $\mathcal{L}$  can bring together any two points  $p_i, p_j \in P$  not separated from each other within the disk boundary by other points of  $P$ . Next, the second limit of  $\mathcal{L}$  can bring together any two points  $p_k, p_l \in P \setminus \{p_i, p_j\}$  not separated from each other within the disk boundary by other points of  $P \setminus \{p_i, p_j\}$ , and so on. It is easy to see that if the limits of  $\mathcal{L}$  are ordered this way, then the points of  $P$  can be joined with  $N$  non-intersecting arcs in the disk, where the endpoints of the  $j$ th arc are the two points in  $P$  brought together by the  $j$ th limit of  $\mathcal{L}$ . We call this diagram an *interior arc connectivity diagram*, and we imagine that the  $j$ th limit of  $\mathcal{L}$  contracts the  $j$ th arc of this diagram to a point (figure 8).

Now two natural questions arise. First, does an interior arc connectivity diagram always give rise to an element of  $\mathcal{S}_N^*$ ? And second, if two mappings  $\mathcal{L}, \mathcal{L}' \in \mathcal{S}_N^*$  constructed according to the previous paragraph, share an interior arc connectivity diagram (thus they are distinguished only by the ordering of their limits), then does  $\mathcal{L}'F = \mathcal{L}F$  for all  $F \in \mathcal{S}_N$ ? In anticipation of an affirmative answer to both questions, it is more convenient to use the interior arc connectivity diagrams to formally state the restrictions on the ordering of the limits in  $\mathcal{L}$ . Because the points brought together by the limits of  $\mathcal{L}$  are in the real axis, we consider these diagrams in the upper half-plane first.

**Definition 6.** Let  $x_1 < x_2 < \dots < x_{2N}$  be the coordinates of a point  $\mathbf{x} \in \Omega_0$ . An *upper half-plane arc connectivity diagram* on  $x_1, \dots, x_{2N}$  is a collection of  $N$  curves, called *arcs*, in the upper half-plane such that

- each arc has its two endpoints among  $x_1, \dots, x_{2N}$ ,
- the two endpoints of each arc are distinct,
- no point among  $x_1, \dots, x_{2N}$  is an endpoint of two different arcs,
- no two different arcs intersect each other, and no arc intersects itself.

There are  $C_N$  topologically distinct upper half-plane arc connectivity diagrams on  $x_1, \dots, x_{2N}$  [43], where  $C_N$  is the  $N$ th Catalan number (4).

**Definition 7.** After we enumerate all  $C_N$  of the upper half-plane arc connectivity diagrams on  $x_1, \dots, x_{2N}$ , let the  $k$ th *upper half-plane arc connectivity diagram* be the  $k$ th of these. Furthermore, let the  $k$ th *polygon interior arc connectivity diagram* be the image of the  $k$ th diagram under a conformal bijection that sends the upper half-plane

onto the interior of a  $2N$ -sided regular polygon  $\mathcal{P}$ , with the coordinates  $x_1, \dots, x_{2N}$  sent to the vertices of  $\mathcal{P}$ . We call either diagram the  $k$ th interior arc connectivity diagram (or more concisely, the  $k$ th connectivity diagram).

**Notation 8.** For any  $a < b \in \mathbb{R}$ , let  $(b, a)$  be the complement of  $[a, b]$  in the one-point compactification of  $\mathbb{R}$ .

**Definition 9.** Let  $x_1 < x_2 < \dots < x_{2N}$  be the coordinates of a point  $\mathbf{x} \in \Omega_0$ , choose one of the  $C_N$  available half-plane arc connectivity diagrams on  $x_1, \dots, x_{2N}$ , enumerate the  $N$  arcs of this diagram in some arbitrary way, and let  $x_{i_{2j-1}} < x_{i_{2j}}$  be the endpoints of the  $j$ th arc.

I Let  $\pi$  be a projection that removes all coordinates of  $\mathbf{x} \in \Omega_0$  that are in  $(x_{i_{2j-1}}, x_{i_{2j}})$  and some even number of coordinates of  $\mathbf{x}$  that are in  $(x_{i_{2j}}, x_{i_{2j-1}})$ . If  $\pi$  removes  $2M$  coordinates with  $M \in \{1, \dots, N\}$ , then let  $F \in \mathcal{S}_{N-M}$  and  $\kappa \in (0, 8)$ . We define  $\bar{\ell}_j : \mathcal{S}_{N-M} \rightarrow \mathcal{S}_{N-M-1}$  by

$$\bar{\ell}_j F(\pi_{i_{2j-1}, i_{2j}} \circ \pi(\mathbf{x})) := \lim_{x_{i_{2j}} \rightarrow x_{i_{2j-1}}} (x_{i_{2j}} - x_{i_{2j-1}})^{6/\kappa-1} F(\pi(\mathbf{x})). \quad (85)$$

Lemma 4 guarantees that this limit exists and does not depend on  $x_{i_{2j-1}}$ , and lemma 5 guarantees that this limit is in  $\mathcal{S}_{N-M-1}$ .

II Or let  $\pi$  be a projection that removes all coordinates of  $\mathbf{x} \in \Omega_0$  that are in  $(x_{i_{2j}}, x_{i_{2j-1}})$  and some even number of coordinates of  $\mathbf{x}$  that are in  $(x_{i_{2j-1}}, x_{i_{2j}})$ . If  $\pi$  removes  $2M$  coordinates with  $M \in \{1, \dots, N\}$ , then let  $F \in \mathcal{S}_{N-M}$  and  $\kappa \in (0, 8)$ . We define  $\underline{\ell}_j : \mathcal{S}_{N-M} \rightarrow \mathcal{S}_{N-M-1}$  by

$$\underline{\ell}_j F(\pi_{i_{2j-1}, i_{2j}} \circ \pi(\mathbf{x})) := \lim_{t \rightarrow \infty} (2t)^{6/\kappa-1} F(\pi(\mathbf{x}))|_{(x_{i_{2j-1}}, x_{i_{2j}}) = (-t, t)}. \quad (86)$$

Lemma 4 guarantees that this limit exists, and lemma 5 guarantees that this limit is in  $\mathcal{S}_{N-M-1}$ .

Let  $F \in \mathcal{S}_N$ ,  $M \in \{1, \dots, N\}$ , and  $\kappa \in (0, 8)$ . Enumerate the  $N$  arcs of the same connectivity diagram, with the index  $k \in \{1, \dots, N\}$  labeling the arc with endpoints at  $x_{i_{2j-1}}$  and  $x_{i_{2j}}$  such that if we let  $j_k = j$ , then exactly one of the following is true for each  $k \in \{1, \dots, N\}$ :

1.  $\bigcup_{l > k}^N \{x_{i_{2j_l-1}}, x_{i_{2j_l}}\} \subset (x_{i_{2j_k-1}}, x_{i_{2j_k}})$ , or
2.  $\bigcup_{l > k}^N \{x_{i_{2j_l-1}}, x_{i_{2j_l}}\} \subset (x_{i_{2j_k}}, x_{i_{2j_k-1}})$ .

Then  $\mathcal{L} : \mathcal{S}_N \rightarrow \mathcal{S}_{N-M}$  defined by  $\mathcal{L}F := \ell_{j_M} \ell_{j_{M-1}} \dots \ell_{j_2} \ell_{j_1} F$  is an allowable sequence of  $M$  limits involving the coordinates  $x_{i_{2j_1-1}}, x_{i_{2j_1}}, \dots, x_{i_{2j_M-1}}, x_{i_{2j_M}}$ . In this definition for  $\mathcal{L}$ ,

- if item 1 above is true for a particular  $k \in \{1, \dots, N\}$ , then  $\ell_{j_k} = \underline{\ell}_{j_k}$  in the definition of  $\mathcal{L}$ ,
- if item 2 above is true for a particular  $k \in \{1, \dots, N\}$ , then  $\ell_{j_k} = \bar{\ell}_{j_k}$  in the definition of  $\mathcal{L}$ .

If  $M = N$ , then the aforementioned rule does not specify whether  $\ell_{j_N} = \bar{\ell}_{j_N}$  or  $\ell_{j_N} = \underline{\ell}_{j_N}$ . But also if  $M = N$ , then  $\ell_{j_N}$  acts on an element of  $\mathcal{S}_1$ . According to (16), this element is a multiple of

$$F(x_{i_{2N-1}}, x_{i_{2N}}) = (x_{i_{2N}} - x_{i_{2N-1}})^{1-6/\kappa}, \quad (87)$$

the image of which under  $\bar{\ell}_{j_N}$  equals its image under  $\underline{\ell}_{j_N}$ . Thus, we may take either  $\ell_{j_N} = \bar{\ell}_{j_N}$  or  $\ell_{j_N} = \underline{\ell}_{j_N}$  in  $\mathcal{L}$ . Finally, we let the half-plane (resp. polygon) arc connectivity diagram (or more concisely, the half-plane (resp. polygon) diagram) for  $\mathcal{L}$  be the upper half-plane (resp. polygon interior) arc connectivity diagram with the arcs whose endpoints are among

$$\{x_{i_{2j_{M+1}-1}}, x_{i_{2j_{M+1}}}, \dots, x_{i_{2j_N-1}}, x_{i_{2j_N}}\} \quad (88)$$

deleted. We call either diagram the arc connectivity diagram for  $\mathcal{L}$ .

The part of definition 9 that defines an allowable sequence of  $M$  limits anticipates the following lemma.

**Lemma 10.** Suppose that  $F \in \mathcal{S}_N$  and  $\kappa \in (0, 8)$ , and let  $\mathcal{L} = \ell_{j_M} \ell_{j_{M-1}} \dots \ell_{j_2} \ell_{j_1}$  be an allowable sequence of  $M \leq N$  limits. Then the limit  $\mathcal{L}F$  exists and is in  $\mathcal{S}_{N-M}$ . In particular, if  $M = N$ , then the limit is a real number.

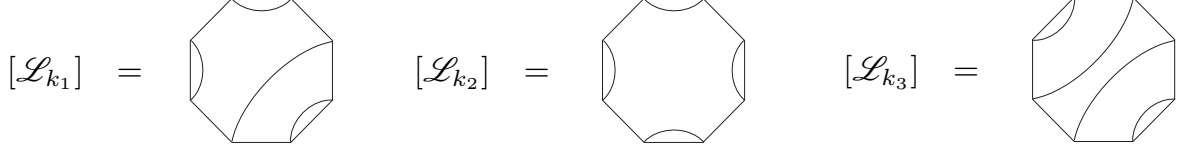


FIG. 9: Polygon diagrams for three different equivalence classes of allowable sequences of  $N = 4$  limits. The other  $C_4 - 3 = 11$  diagrams are found by rotating one of these three.

*Proof.* Items 1–2 in definition 9 imply that the two coordinates  $x_{i_{2j_1-1}}$  and  $x_{i_{2j_1}}$  satisfy either

- $x_{i_{2j_1-1}} = x_i$  and  $x_{i_{2j_1}} = x_{i+1}$  for some  $i \in \{1, \dots, 2N-1\}$ , and therefore  $\ell_{j_1} = \bar{\ell}_{j_1}$ , or
- $x_{i_{2j_1-1}} = x_1$  and  $x_{i_{2j_1}} = x_{2N}$ , and therefore  $\ell_{j_1} = \underline{\ell}_{j_1}$ .

In either case, lemmas 4–5 guarantee that the limit  $F_0 := \ell_{j_1} F$  exists and is in  $\mathcal{S}_{N-1}$ . Furthermore,  $\ell_{j_M} \ell_{j_{M-1}} \dots \ell_{j_2}$  is, according to definition 9, an allowable sequence of limits involving the coordinates  $x_{i_{2j_2-1}}, x_{i_{2j_2}}, \dots, x_{i_{2j_M-1}}, x_{i_{2j_M}}$ . Therefore, items 1–2 in definition 9 imply that the two coordinates  $x_{i_{2j_2-1}}$  and  $x_{i_{2j_2}}$  satisfy either

- item 1 in definition 9, so  $\ell_{j_1} = \bar{\ell}_{j_1}$  and there are no points among  $x_{i_{2j_3-1}}, x_{i_{2j_3}}, \dots, x_{i_{2j_M-1}}, x_{i_{2j_M}}$  in  $(x_{i_{2j_2-1}}, x_{i_{2j_2}})$ , or
- item 2 in definition 9, so  $\ell_{j_1} = \underline{\ell}_{j_1}$  and there are no points among  $x_{i_{2j_3-1}}, x_{i_{2j_3}}, \dots, x_{i_{2j_M-1}}, x_{i_{2j_M}}$  in  $(x_{i_{2j_2}}, x_{i_{2j_2-1}})$ .

In either case, lemma 4 guarantees that the limit  $\ell_{j_2} F_0 = \ell_{j_2} \ell_{j_1} F$  exists, and lemma 5 guarantees that it is in  $\mathcal{S}_{N-2}$ . We repeat this reasoning  $M-2$  more times to prove the lemma.  $\square$

The vector space of allowable sequences of limits acting on  $\mathcal{S}_N$  has a natural partition into equivalence classes given by the following definition. (See figure 9 for an example.)

**Definition 11.** Two allowable sequences of  $M \in \{1, \dots, N\}$  limits  $\mathcal{L} = \ell_{j_M} \dots \ell_{j_1}$  and  $\mathcal{L}' = \ell_{j'_M} \dots \ell_{j'_1}$  involving the same coordinates of  $\mathbf{x} \in \Omega_0$  are *equivalent* if their arc connectivity diagrams are identical. This defines an equivalence relation on the set of all allowable sequences of  $M$  limits. Let  $[\mathcal{L}]$  be the equivalence class containing  $\mathcal{L}$ . If  $M = N$ , then let  $\mathcal{L}_k$  stand for an allowable sequence of  $N$  limits whose arc connectivity diagram is the  $k$ th connectivity diagram, and let  $\mathcal{B}_N^* := \{[\mathcal{L}_1], [\mathcal{L}_2], \dots, [\mathcal{L}_{C_N}]\}$ .

Because there are exactly  $C_N$  interior arc connectivity diagrams, with  $C_N$  the  $N$ th Catalan number (4), it immediately follows that the cardinality of  $\mathcal{B}_N^*$  is  $C_N$ .

**Lemma 12.** Suppose that  $F \in \mathcal{S}_N$  and  $\kappa \in (0, 8)$ , and let  $[\mathcal{L}]$  be an equivalence class of allowable sequences of  $M \in \{1, \dots, N\}$  limits. Then  $[\mathcal{L}]F$  is well-defined in the sense that  $\mathcal{L}'F = \mathcal{L}''F$  for all  $\mathcal{L}', \mathcal{L}'' \in [\mathcal{L}]$ .

*Proof.* If  $N = 1$  or  $2$ , then we can prove the lemma by working directly with the elements of  $\mathcal{S}_N$ , all of which are explicitly known (16, 19). Therefore, we assume that  $N > 2$  throughout this proof.

The proof is by induction on  $M$ . To begin, we suppose that  $M = 2$ , so  $[\mathcal{L}]$  has at most two elements. We further assume that  $[\mathcal{L}]$  has exactly two elements (or else the proof of the case  $M = 2$  is trivial) and the arcs in the half-plane arc connectivity diagram for  $[\mathcal{L}]$  are un-nested. We enumerate these arcs so the left arc has  $j = 1$ , the right arc has  $j = 2$ , and  $[\mathcal{L}] = \{\ell_2 \bar{\ell}_1, \bar{\ell}_1 \ell_2\}$ . Then because the two arcs are un-nested, we have  $x_{i_1} < x_{i_2} < x_{i_3} < x_{i_4}$ . Furthermore, because  $[\mathcal{L}]$  has exactly two elements and the two arcs in its diagram are un-nested, definition 9 implies that the  $2N-4$  coordinates of  $\mathbf{x} \in \Omega_0$  not involved in  $[\mathcal{L}]$  must lie outside of  $(x_{i_1}, x_{i_2}) \cup (x_{i_3}, x_{i_4})$ . Thus, we have  $i_1 = i$ ,  $i_2 = i+1$ ,  $i_3 = j$ , and  $i_4 = j+1$  for some  $i, j \in \{1, \dots, 2N-1\}$  with  $j > i+1$ .

Now, to prove the lemma for the case discussed in the previous paragraph, we must show that  $\bar{\ell}_1 \bar{\ell}_2 F = \bar{\ell}_2 \bar{\ell}_1 F$ , or equivalently,

$$\lim_{x_{i+1} \rightarrow x_i} \lim_{x_{j+1} \rightarrow x_j} (x_{i+1} - x_i)^{6/\kappa-1} (x_{j+1} - x_j)^{6/\kappa-1} F(\mathbf{x}) = \lim_{x_{j+1} \rightarrow x_j} \lim_{x_{i+1} \rightarrow x_i} (x_{i+1} - x_i)^{6/\kappa-1} (x_{j+1} - x_j)^{6/\kappa-1} F(\mathbf{x}). \quad (89)$$

In the only other possible scenario with  $M = 2$  and  $[\mathcal{L}]$  having exactly two elements, one of the two arcs in the half-plane arc connectivity diagram for  $[\mathcal{L}]$  nests the other arc, and the  $2N-4$  coordinates of  $\mathbf{x} \in \Omega_0$  not involved in  $[\mathcal{L}]$  must lie inside the outer arc and outside the inner arc. By conformally transforming via (81) so neither image

of the two arcs nests the other, we can make use of (89) to prove the equivalent statement  $\bar{\ell}_1 \bar{\ell}_2 F = \bar{\ell}_2 \bar{\ell}_1 F$ , where we have labeled the inner and outer arc the first and second arc respectively.

In our proof of (89), we let  $x := x_i$ ,  $y := x_j$ ,  $\delta := x_{i+1} - x_i$ ,  $\epsilon := x_{j+1} - x_j$ , we relabel the other  $2N - 4$  coordinates  $\{x_k\}_{k \neq i, i+1, j, j+1}$  in ascending order by  $\{\xi_1, \dots, \xi_{2N-4}\}$ , and we let  $\xi = (\xi_1, \xi_2, \dots, \xi_{2N-5}, \xi_{2N-4})$ . (This definition of  $\xi$  resembles, but is not the same as, that in the proof of lemmas 3–5). We restrict to  $\delta, \epsilon \in (0, b)$  where  $b$  is small enough to ensure that  $x_{i+1}$  and  $x_{j+1}$  are respectively less than  $x_{i+2}$  and  $x_{j+2}$ . Finally, we let

$$I(\xi; x, \delta; y, \epsilon) = \epsilon^{6/\kappa-1} \delta^{6/\kappa-1} F(\xi_1, \dots, \xi_{i-1}, x, x + \delta, \xi_i, \dots, \xi_{j-3}, y, y + \epsilon, \xi_{j-2}, \dots, \xi_{2N-4}), \quad (90)$$

and when expressed in terms of these quantities, (89) becomes

$$\lim_{\delta \downarrow 0} \lim_{\epsilon \downarrow 0} I(\xi; x, \delta; y, \epsilon) = \lim_{\epsilon \downarrow 0} \lim_{\delta \downarrow 0} I(\xi; x, \delta; y, \epsilon), \quad (91)$$

which we wish to prove. Now, the integral equation (60) expressed in terms of  $I$ ,  $\xi$ ,  $x$ ,  $\delta$ ,  $y$ , and  $\epsilon$  becomes

$$I(\xi; x, \delta; y, \epsilon) = I(\xi; x, b; y, \epsilon) - \frac{\kappa}{4} J(\delta, a) \partial_\delta I(\xi; x, b; y, \epsilon) + \int_\delta^b J(\delta, \eta) \mathcal{N}[I](\xi; x, \eta; y, \epsilon) d\eta, \quad 0 < \delta < b, \quad (92)$$

where  $J$  is the Green function (61) and  $\mathcal{N}$  is the differential operator

$$\begin{aligned} \mathcal{N} := & \frac{\partial_x}{\eta} - \sum_k \left( \frac{\partial_k}{\xi_k - x - \eta} - \frac{(6 - \kappa)/2\kappa}{(\xi_k - x - \eta)^2} \right) - \frac{\partial_y}{y - x - \eta} + \frac{(6 - \kappa)/2\kappa}{(y - x - \eta)^2} \\ & + \frac{\epsilon \partial_\epsilon}{(y - x - \eta)(y + \epsilon - x - \eta)} + \frac{(6 - \kappa)/2\kappa}{(y + \epsilon - x - \eta)^2} + \frac{1 - 6/\kappa}{(y - x - \eta)(y + \epsilon - x - \eta)}. \end{aligned} \quad (93)$$

With  $x$ ,  $y$ , and the coordinates of  $\xi$  fixed to distinct values, we prove (91) by showing that  $I(\xi; x, \delta; y, \epsilon)$  approaches its limit  $I(\xi; x, 0; y, \epsilon)$  as  $\delta \downarrow 0$  (guaranteed to exist by lemma 3) uniformly over  $0 < \epsilon < b$ . Equation (92) with  $\delta$  replaced by zero and then  $b$  replaced by  $\delta$  gives (using the bound on  $J$  mentioned just below (61))

$$\sup_{0 < \epsilon < b} |I(\xi; x, \delta; y, \epsilon) - I(\xi; x, 0; y, \epsilon)| \leq \frac{\kappa}{8 - \kappa} \sup_{0 < \epsilon < b} |\delta \partial_\delta I(\xi; x, \delta; y, \epsilon)| + \frac{4}{8 - \kappa} \int_0^\delta \sup_{0 < \epsilon < b} |\eta \mathcal{N}[I](\xi; x, \eta; y, \epsilon)| d\eta, \quad (94)$$

so to prove uniformness, it suffices to show that the right side of (94) vanishes as  $\delta \downarrow 0$ .

First, we prove that the integral in (94) vanishes by showing that its integrand is bounded over  $0 < \eta < b$ . The proof of this statement resembles the proof of lemma 3, but it has a key difference. In the proof of lemma 3, constants arising from the Schauder interior estimate (50) grow without bound as  $\epsilon \downarrow 0$  because the coefficients of the elliptic PDE (49) (recast in terms of the variables used in this proof) grow without bound in this limit. To avoid this issue, we construct a new elliptic PDE whose coefficients are bounded as  $\epsilon \downarrow 0$ .

For  $k \neq i, i + 1, j$ , or  $j + 1$ , the null-state PDE centered on  $x_k$  becomes (now with  $k \in \{1, \dots, 2N - 4\}$ )

$$\begin{aligned} & \left[ \frac{\kappa}{4} \partial_k^2 + \sum_{l \neq k} \left( \frac{\partial_l}{\xi_l - \xi_k} - \frac{(6 - \kappa)/2\kappa}{(\xi_l - \xi_k)^2} \right) \right. \\ & + \frac{\partial_x}{x - \xi_k} - \frac{\delta \partial_\delta}{(x - \xi_k)(x + \delta - \xi_k)} + \frac{6/\kappa - 1}{(x - \xi_k)(x + \delta - \xi_k)} - \frac{(6 - \kappa)/2\kappa}{(x - \xi_k)^2} - \frac{(6 - \kappa)/2\kappa}{(x + \delta - \xi_k)^2} \\ & \left. + \frac{\partial_y}{y - \xi_k} - \frac{\epsilon \partial_\epsilon}{(y - \xi_k)(y + \epsilon - \xi_k)} + \frac{6/\kappa - 1}{(y - \xi_k)(y + \epsilon - \xi_k)} - \frac{(6 - \kappa)/2\kappa}{(y - \xi_k)^2} - \frac{(6 - \kappa)/2\kappa}{(y + \epsilon - \xi_k)^2} \right] I(\xi; x, \delta; y, \epsilon) = 0, \end{aligned} \quad (95)$$

the null-state PDE centered on  $x_i$  becomes

$$\begin{aligned} & \left[ \frac{\kappa}{4} (\partial_x - \partial_\delta)^2 + \frac{\partial_\delta}{\delta} + \frac{(6 - \kappa)(\partial_x - \partial_\delta)}{2\delta} + \sum_l \left( \frac{\partial_l}{\xi_l - x} - \frac{(6 - \kappa)/2\kappa}{(\xi_l - x)^2} \right) \right. \\ & \left. + \frac{\partial_y}{y - x} - \frac{\epsilon \partial_\epsilon}{(y - x)(y + \epsilon - x)} - \frac{(6 - \kappa)/2\kappa}{(y - x)^2} - \frac{(6 - \kappa)/2\kappa}{(y + \epsilon - x)^2} + \frac{6/\kappa - 1}{(y - x)(y + \epsilon - x)} \right] I(\xi; x, \delta; y, \epsilon) = 0, \end{aligned} \quad (96)$$



the null-state PDE centered on  $x_{i+1}$  becomes

$$\left[ \frac{\kappa}{4} \partial_\delta^2 - \frac{(\partial_x - \partial_\delta)}{\delta} - \frac{(6 - \kappa) \partial_\delta}{2\delta} + \sum_k \left( \frac{\partial_l}{\xi_l - x - \delta} - \frac{(6 - \kappa)/2\kappa}{(\xi_l - x - \delta)^2} \right) \right. \\ \left. + \frac{\partial_y}{y - x} - \frac{\epsilon \partial_\epsilon}{(y - x)(y + \epsilon - x)} - \frac{(6 - \kappa)/2\kappa}{(y - x)^2} - \frac{(6 - \kappa)/2\kappa}{(y + \epsilon - x)^2} + \frac{6/\kappa - 1}{(y - x)(y + \epsilon - x)} \right] I(\boldsymbol{\xi}; x, \delta; y, \epsilon) = 0, \quad (97)$$

and the null-state PDEs centered on  $x_j$  and  $x_{j+1}$  are found by replacing  $(x, \delta; y, \epsilon) \mapsto (y, \epsilon; x, \delta)$  in (96) and (97) respectively. Also, the three Ward identities (11) become

$$\left[ \sum_k \partial_k + \partial_x + \partial_y \right] I(\boldsymbol{\xi}; x, \delta; y, \epsilon) = 0, \quad (98)$$

$$\left[ \sum_k (\xi_k \partial_k + (6 - \kappa)/2\kappa) + x \partial_x + \delta \partial_\delta + y \partial_y + \epsilon \partial_\epsilon \right] I(\boldsymbol{\xi}; x, \delta; y, \epsilon) = 0, \quad (99)$$

$$\left[ \sum_k (\xi_k^2 \partial_k + (6 - \kappa) \xi_k / \kappa) + x^2 \partial_x + (2x + \delta) \delta \partial_\delta + y^2 \partial_y + (2y + \epsilon) \epsilon \partial_\epsilon \right] I(\boldsymbol{\xi}; x, \delta; y, \epsilon) = 0. \quad (100)$$

The last two identities will prove most useful if we isolate  $\delta \partial_\delta I$  and  $\epsilon \partial_\epsilon I$  in terms of  $I$  and its derivatives with respect to  $x, y$ , and the coordinates of  $\boldsymbol{\xi}$ . We find

$$\delta \partial_\delta I(\boldsymbol{\xi}; x, \delta; y, \epsilon) = \frac{1}{2(x - y) + \delta - \epsilon} \sum_k \left[ (2y + \epsilon - \xi_k) \xi_k \partial_k + (6 - \kappa)(2y + \epsilon - 2\xi_k)/2\kappa \right] I(\boldsymbol{\xi}; x, \delta; y, \epsilon) \\ + \frac{1}{2(x - y) + \delta - \epsilon} \left[ (2y + \epsilon - x) x \partial_x + (y + \epsilon) y \partial_y \right] I(\boldsymbol{\xi}; x, \delta; y, \epsilon), \quad (101)$$

$$\epsilon \partial_\epsilon I(\boldsymbol{\xi}; x, \delta; y, \epsilon) = \frac{1}{2(y - x) + \epsilon - \delta} \sum_k \left[ (2x + \delta - \xi_k) \xi_k \partial_k + (6 - \kappa)(2x + \delta - 2\xi_k)/2\kappa \right] I(\boldsymbol{\xi}; x, \delta; y, \epsilon) \\ + \frac{1}{2(y - x) + \epsilon - \delta} \left[ (x + \delta) x \partial_x + (2x + \delta - y) y \partial_y \right] I(\boldsymbol{\xi}; x, \delta; y, \epsilon). \quad (102)$$

We use PDEs (96–100) to construct a new PDE that has  $x, y$ , and the coordinates of  $\boldsymbol{\xi}$  as independent variables, that has  $\delta$  and  $\epsilon$  as parameters, and that is strictly elliptic in an arbitrary compactly embedded subset of  $\pi_{i+1, j+1}(\Omega_0)$ . To begin, we subtract (97) from (96) and multiply the result by  $\delta$  to find

$$\left[ \frac{\kappa}{4} \delta \partial_x^2 - \frac{\kappa}{2} \partial_x \delta \partial_\delta + \frac{8 - \kappa}{2} \partial_x - \sum_k \left( \frac{\delta^2 \partial_k}{(\xi_k - x)(\xi_k - x - \delta)} + \frac{[\delta + 2(x - \xi_k)] \delta^2 (6 - \kappa)/2\kappa}{(\xi_k - x)^2 (\xi_k - x - \delta)^2} \right) \right. \\ - \frac{\delta^2 \partial_y}{(y - x)(y - x - \delta)} - \frac{[\delta + 2(x - y)] \delta^2 (6 - \kappa)/2\kappa}{(y - x)^2 (y - x - \delta)^2} - \frac{\delta^2 (2x - 2y + \delta - \epsilon) \epsilon \partial_\epsilon}{(y - x)(y - x - \delta)(y + \epsilon - x)(y + \epsilon - x - \delta)} \\ \left. - \frac{\delta^2 (2x - 2y + \delta - \epsilon)(1 - 6/\kappa)}{(y - x)(y - x - \delta)(y + \epsilon - x)(y + \epsilon - x - \delta)} - \frac{[\delta + 2(x - y - \epsilon)] \delta^2 (6 - \kappa)/2\kappa}{(y + \epsilon - x)^2 (y + \epsilon - x - \delta)^2} \right] I(\boldsymbol{\xi}; x, \delta; y, \epsilon) = 0. \quad (103)$$

Next, we use (98) to eliminate  $\partial_y I$  from (101), and we insert the result into (103) to generate a PDE whose principal part only contains  $\partial_x^2 I$  and the mixed partial derivatives  $\partial_x \partial_k I$  for  $k \in \{1, \dots, 2N - 4\}$ . The coefficient of the former term in the principal part is

$$a(x, \delta; y, \epsilon) = \frac{\kappa}{4} \delta + \frac{\kappa}{2} \left( \frac{(x - y)^2 - \epsilon(x - y)}{2(x - y) + \delta - \epsilon} \right), \quad (104)$$

and we note that  $a$  restricted to a compactly embedded subset of  $\pi_{i+1, j+1}(\Omega_0)$  (so that  $x - y$  is bounded away from zero) does not vanish or grow without bound as  $\epsilon \downarrow 0$  or  $\delta \downarrow 0$ . The other coefficients in the principle part of this PDE exhibit this property too, and none of the other coefficients in this PDE grow without bound as  $\epsilon \downarrow 0$  or  $\delta \downarrow 0$  either. Next, by replacing  $(x, \delta; y, \epsilon) \mapsto (y, \epsilon; x, \delta)$ , we generate another PDE whose principal part only contains  $\partial_y^2 I$  (with coefficient  $a(y, \epsilon; x, \delta)$ ) and the mixed partial derivatives  $\partial_y \partial_k I$  for  $k \in \{1, \dots, 2N - 4\}$ , and with the aforementioned features of the companion PDE that generated it.

Next, we take a linear combination of the two PDEs that we constructed in the previous paragraph, with respective nonzero coefficients  $a(x, \delta; y, \epsilon)^{-1}$  and  $a(y, \epsilon; x, \delta)^{-1}$ , and the  $2N - 4$  null-state PDEs in (95), each with the same nonzero coefficient  $4c/\kappa$  with  $c \in \mathbb{R}$ , to find a new PDE whose principal part only contains  $\partial_x^2 I$ ,  $\partial_y^2 I$ ,  $\partial_k^2 I$ , and the mixed partial derivatives  $\partial_x \partial_k I$  and  $\partial_y \partial_k I$  with  $k \in \{1, \dots, 2N - 4\}$ . Furthermore, we can use (101–102) again to replace the first derivatives  $\delta \partial_\delta I$  and  $\epsilon \partial_\epsilon I$  in this current PDE with linear combinations of first derivatives in  $x$ ,  $y$ , and the coordinates of  $\xi$ . This produces a final PDE (which is very complicated, so we do not display it here) for which  $x$ ,  $y$ , and the coordinates of  $\xi$  are independent variables while  $\delta$  and  $\epsilon$  are simply parameters, the coefficients in the principle part of this final PDE do not vanish or grow without bound as  $\epsilon \downarrow 0$  or  $\delta \downarrow 0$ , and none of the coefficients of the other terms grow without bound as  $\epsilon \downarrow 0$  or  $\delta \downarrow 0$ .

Finally, we argue that in any open set  $\mathcal{U}_0 \subset \subset \pi_{i+1, j+1}(\Omega_0)$ , there exists a choice for  $c$  such that the PDE constructed in the previous paragraph is strictly elliptic in that open set. The coefficient matrix for its principal part is

$$\begin{matrix} & \xi_1 & \xi_2 & \dots & \xi_{2N-5} & \xi_{2N-4} & x & y \\ \xi_1 & c & 0 & \dots & 0 & 0 & a_{x,1} & a_{y,1} \\ \xi_2 & 0 & c & & & 0 & a_{x,2} & a_{y,2} \\ \vdots & \vdots & & \ddots & & \vdots & \vdots & \vdots \\ \xi_{2N-5} & 0 & & & c & 0 & a_{x,2N-5} & a_{y,2N-5} \\ \xi_{2N-4} & 0 & 0 & \dots & 0 & c & a_{x,2N-4} & a_{y,2N-4} \\ x & a_{x,1} & a_{x,2} & \dots & a_{x,2N-5} & a_{x,2N-4} & 1 & 0 \\ y & a_{y,1} & a_{y,2} & \dots & a_{y,2N-5} & a_{y,2N-4} & 0 & 1 \end{matrix} \quad (105)$$

where  $a_{x,k}$  (resp.  $a_{y,k}$ ) is half of the coefficient of  $\partial_x \partial_k I$  (resp.  $\partial_y \partial_k I$ ). This matrix is positive definite if all of its leading principal minors are positive. If  $c > 0$ , then the determinant of the first  $2N - 4$  leading principal minors are evidently positive. We find the remaining two principal minors by using the determinant formula

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \implies \det M = \det A \det(D - CA^{-1}B), \quad (106)$$

where  $A$  and  $D$  are square blocks of the matrix  $M$ , and where  $B$  and  $C$  are blocks that fill the part of  $M$  above  $D$  and beneath  $A$  respectively. Using this formula, we find that the  $(2N - 3)$ th leading principal minor of the coefficient matrix (105) is

$$c^{2N-4} \left( 1 - \frac{|a_x|^2}{c^{2N-4}} \right), \quad (107)$$

where  $a_x$  is the vector in  $\mathbb{R}^{2N-4}$  whose  $k$ th entry is  $a_{x,k}$ , and the  $(2N - 2)$ th leading principal minor (this is simply the determinant of (105)) is

$$c^{2N-4} \det \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{c^{2N-4}} \begin{pmatrix} a_x \cdot a_x & a_x \cdot a_y \\ a_y \cdot a_x & a_y \cdot a_y \end{pmatrix} \right], \quad (108)$$

where  $a_y$  is the vector in  $\mathbb{R}^{2N-4}$  whose  $k$ th entry is  $a_{y,k}$ . By choosing  $c$  sufficiently large, we can ensure that determinants (107) and (108) are greater than, say, one at all points in an open set  $\mathcal{U}_0 \subset \subset \pi_{i+1, j+1}(\Omega_0)$  and for all  $(\delta, \epsilon) \in (0, b) \times (0, b)$ . Because all of the components of  $a_x$  and  $a_y$  are bounded on  $\mathcal{U}_0$ , the eigenvalues of (105) are also bounded on  $\mathcal{U}_0$ . This fact and the fact that the product of these eigenvalues (equaling the determinant of (105)) is greater than one on  $\mathcal{U}_0$  imply that all of these eigenvalues are bounded away from zero over this set. Thus, the PDE that we have constructed is strictly elliptic in  $\mathcal{U}_0$ .

The existence of a PDE governing  $I$  that is strictly elliptic in  $\mathcal{U}_0$  implies the Schauder interior estimate [22]

$$\sum_k d \sup_{\mathcal{U}_1} |\partial_k I(\xi; x, \delta; y, \epsilon)| + d \sup_{\mathcal{U}_1} |\partial_x I(\xi; x, \delta; y, \epsilon)| + d \sup_{\mathcal{U}_1} |\partial_y I(\xi; x, \delta; y, \epsilon)| \leq C(\delta, \epsilon) \sup_{\mathcal{U}_0} |I(\xi; x, \delta; y, \epsilon)|, \quad (109)$$

where  $\mathcal{U}_1 \subset \subset \mathcal{U}_0$  is an open set,  $d = \text{dist}(\partial \mathcal{U}_0, \partial \mathcal{U}_1)$ , and  $C$  is some function. The Ward identities (101–102) combined with the estimate (109) further imply

$$d \sup_{\mathcal{U}_1} |\delta \partial_\delta I(\xi; x, \delta; y, \epsilon)| + d \sup_{\mathcal{U}_1} |\epsilon \partial_\epsilon I(\xi; x, \delta; y, \epsilon)| \leq C'(\delta, \epsilon) \sup_{\mathcal{U}_0} |I(\xi; x, \delta; y, \epsilon)|, \quad (110)$$

with  $C'$  another function. Because the coefficients of this strictly elliptic PDE are bounded as  $\delta \downarrow 0$  and  $\epsilon \downarrow 0$ , it follows that  $C$  and  $C'$  are bounded by a constant  $C''$  for all  $(\epsilon, \delta) \in (0, b) \times (0, b)$ . After applying this fact to the

estimates (109–110) and inserting the result into the supremum of (92) over  $\mathcal{U}_1$ , (92) gives

$$\sup_{\mathcal{U}_1} |I(\xi; x, \delta; y, \epsilon)| \leq c_1 \sup_{\mathcal{U}_1} |I(\xi; x, b; y, \epsilon)| + c_2 \int_{\delta}^b \sup_{\mathcal{U}_0} |I(\xi'; x', \eta; y', \epsilon)| d\eta \quad (111)$$

where  $c_1$  and  $c_2$  are positive constants.

Now we use (111) to show that  $I(\xi; x, \delta; y, \epsilon)$  is bounded over  $\{(\epsilon, \delta) \mid 0 < \epsilon, \delta < b\}$ . The bound (20) gives

$$\sup_{\mathcal{U}_0} |I(\xi; x, \delta, y, \epsilon)| \leq c_3 \delta^{-p} \epsilon^{-p}, \quad 0 < \delta, \epsilon < b. \quad (112)$$

We assume that  $p$  is positive; otherwise the proof is trivial. By inserting this estimate into (111), using the estimates (109–110), and integrating, we easily find that the supremum of  $I(\xi; x, \delta, y, \epsilon)$  over  $\mathcal{U}_1$  is  $O(\delta^{-p+1} \epsilon^p)$ . By repeating this process another  $p - 1$  times, as in the proof of lemma 3, we eventually find an open set  $\mathcal{U}_p \subset \subset \pi_{i+1, j+1}(\Omega_0)$  and a constant  $c_{p+3}$  for which

$$\sup_{\mathcal{U}_p} |I(\xi; x, \delta, y, \epsilon)| \leq c_{p+3} \epsilon^{-p}, \quad 0 < \delta < b. \quad (113)$$

Now we insert the estimate (113) into the integral equation generated by the replacements  $(x, \delta; y, \epsilon) \mapsto (y, \epsilon; x, \delta)$ ,  $\mathcal{U}_0 \mapsto \mathcal{U}_p$ , and  $\mathcal{U}_1 \mapsto \mathcal{U}_{p+1} \subset \subset \mathcal{U}_p$  in (111), and we repeat these steps in the switched variables to eventually find an open set  $\mathcal{U}_{2p} \subset \subset \pi_{i+1, j+1}(\Omega_0)$  such that the supremum of  $|I(\xi; x, \delta, y, \epsilon)|$  over  $\mathcal{U}_{2p}$  is bounded over  $\{(\epsilon, \delta) \mid 0 < \epsilon, \delta < b\}$ . This fact and estimates (109–110) imply that the integrand of (94) is a bounded function of  $\eta$ . Hence, the second term on the right side of (94) vanishes as  $\delta \downarrow 0$ .

Now we argue that the first term on the right side of (94) vanishes as  $\delta \downarrow 0$ . After re-expressing (63) in terms of the quantities used in this proof, we take the supremum of both sides over  $0 < \epsilon < b$  to find

$$\sup_{0 < \epsilon < b} |\delta \partial_{\delta} I(\xi; x, \delta; y, \epsilon)| = \left(\frac{\delta}{b}\right)^{8/\kappa-1} \sup_{0 < \epsilon < b} |b \partial_{\delta} I(\xi; x, b; y, \epsilon)| + \frac{4}{\kappa} \int_{\delta}^b \left(\frac{\delta}{\eta}\right)^{8/\kappa-1} \sup_{0 < \epsilon < b} |\eta \mathcal{N}[I](\xi; x, \eta; y, \epsilon)| d\eta \quad (114)$$

for  $0 < \delta < b$ . We just argued that the supremum in the integrand of (114) is bounded over  $0 < \eta < b$ . In light of this fact, we can repeat the analysis of (64–65) to show that the supremum of  $|\delta \partial_{\delta} I(\xi; x, \delta; y, \epsilon)|$  over  $0 < \epsilon < b$  goes to zero as  $\delta \downarrow 0$ . Thus, the first term on the right side of (94) vanishes as  $\delta \downarrow 0$ . In the previous paragraph, we proved that the second term on the right side of (94) vanishes too. Hence, we conclude from (94) that  $I(\xi; x, \delta; y, \epsilon)$  converges to its limit as  $\delta \downarrow 0$  uniformly over  $0 < \epsilon < b$ . This fact proves the equality supposed in (91) and also (89) when we recall definition (90). Consequently, we have shown that if each element of  $[\mathcal{L}]$  has two limits, then  $[\mathcal{L}]F$  is well-defined for all  $F \in \mathcal{S}_N$ .

Now we suppose that for all  $F \in \mathcal{S}_N$  and all  $[\mathcal{L}']$  whose elements have less than  $M$  limits,  $[\mathcal{L}']F$  is well-defined. We choose an arbitrary  $F \in \mathcal{S}_N$  and an  $[\mathcal{L}]$  involving  $M$  limits  $\ell_1, \dots, \ell_M$ . Each element of  $[\mathcal{L}]$  equals  $\ell_m \mathcal{L}_{\bar{m}}$  for some  $m \in \{1, \dots, M\}$  and some allowable sequence  $\mathcal{L}_{\bar{m}}$  of the  $M - 1$  limits of  $\{\ell_j\}_{j \neq m}$ . We let

$$\mathcal{A} := \{m \in \mathbb{Z}^+ \mid \text{there is an element of } [\mathcal{L}] \text{ that executes the particular limit } \ell_m \text{ last}\}. \quad (115)$$

For fixed  $m \in \mathcal{A}$ , all elements of  $[\mathcal{L}]$  of the form  $\ell_m \mathcal{L}_{\bar{m}}$  are equivalent by definition 11, and we denote their equivalence sub-class by  $\ell_m[\mathcal{L}_{\bar{m}}]$ , where  $[\mathcal{L}_{\bar{m}}]$  is the equivalence class for  $\mathcal{L}_{\bar{m}}$ . By the induction hypothesis,  $[\mathcal{L}_{\bar{m}}]F$  is well-defined for all  $m \in \mathcal{A}$ , so to finish the proof, we show that  $\ell_m[\mathcal{L}_{\bar{m}}]F = \ell_n[\mathcal{L}_{\bar{n}}]F$  for each pair  $(m, n) \in \mathcal{A} \times \mathcal{A}$ .

If  $M < N$ , then the condition  $m, n \in \mathcal{A}$  (i.e., we can collapse either the  $m$ th arc or the  $n$ th arc in the half-plane diagram for  $\mathcal{L}$  last) and  $m \neq n$  constrains the possible connectivities of the arcs in the half-plane diagram for  $[\mathcal{L}]$ . (We recall from definition 9 that we have enumerated the  $M$  arcs in this diagram, the limit  $\ell_j$  corresponds to the  $j$ th arc, and the coordinates  $x_{i_{2j-1}} < x_{i_{2j}}$  of  $\mathbf{x} \in \Omega_0$  are the endpoints of the  $j$ th arc.) Indeed, if the  $m$ th arc nests the  $n$ th arc in this diagram, then the following are true.

1. All coordinates of  $\mathbf{x}$  that are not involved in  $[\mathcal{L}]$  reside in  $(x_{i_{2m-1}}, x_{i_{2n-1}}) \cup (x_{i_{2n}}, x_{i_{2m}})$ . Indeed, if one of these coordinates resides in  $(x_{i_{2n-1}}, x_{i_{2n}})$  instead, then  $\ell_n = \underline{\ell}_n$  and  $\ell_m = \underline{\ell}_m$  in all elements of  $[\mathcal{L}]$ . Hence, the limit  $\underline{\ell}_n$  necessarily follows  $\underline{\ell}_m$  in all elements of  $[\mathcal{L}]$ , contradicting the supposition that  $m \in \mathcal{A}$ . A similar argument shows that none of these coordinates reside in  $(x_{i_{2m}}, x_{i_{2m-1}})$  either, thus proving the claim.
2. We have  $\ell_m = \underline{\ell}_m$ , and  $\ell_n = \bar{\ell}_n$  in all elements of  $[\mathcal{L}]$ . Indeed, this follows immediately from item 1 above.
3. No arc in the half-plane diagram for  $[\mathcal{L}]$  simultaneously has one endpoint in  $(x_{i_{2m-1}}, x_{i_{2n-1}})$  and its other endpoint in  $(x_{i_{2n}}, x_{i_{2m}})$ . Indeed, suppose that the contrary is true for the  $j$ th arc. Then according to definition

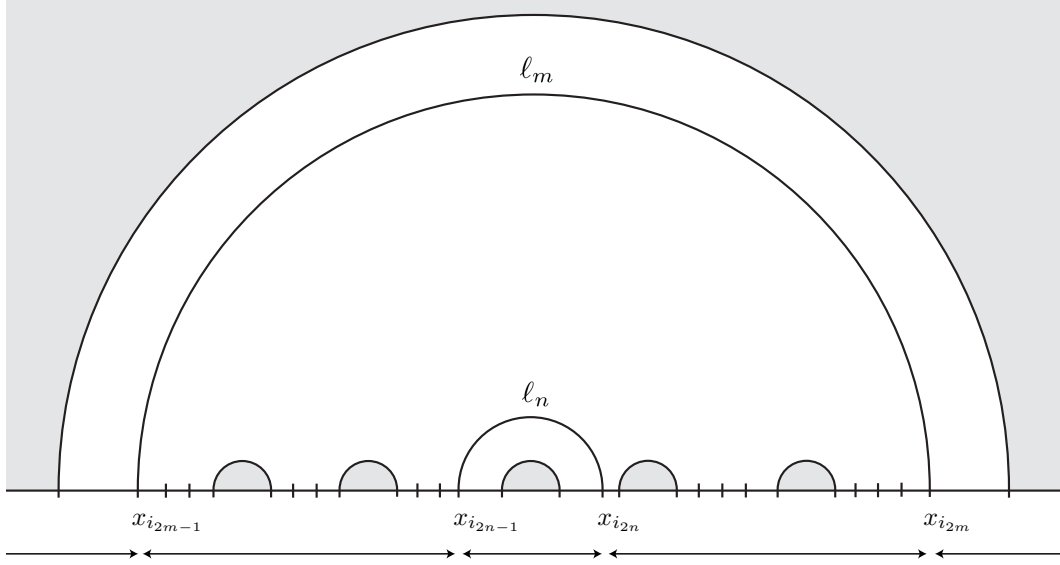


FIG. 10: The half-plane diagram for an  $[\mathcal{L}]$  with  $M$  limits and with  $m, n \in \mathcal{A}$  (115). The  $m$ th and  $n$ th arcs, respectively corresponding to  $\ell_n$  and  $\ell_m$ , are shown while the other  $M - 2$  arcs are not shown but exist only in the gray regions. The tick marks in  $(x_{i_{2m-1}}, x_{i_{2n-1}}) \cup (x_{i_{2n}}, x_{i_{2m}})$  locate the  $2(N - M)$  coordinates of  $\mathbf{x} \in \Omega_0$  not involved in  $[\mathcal{L}]$ .

- 9, either all or none of the coordinates of  $\mathbf{x}$  that are not involved in  $[\mathcal{L}]$  reside between the endpoints of the  $j$ th arc. In the former case,  $\ell_j = \underline{\ell}_j$ , and  $\underline{\ell}_j$  necessarily follows  $\underline{\ell}_m$  in each element of  $[\mathcal{L}]$ , contradicting the supposition that  $m \in \mathcal{A}$ . In the latter case,  $\ell_j = \bar{\ell}_j$ , and  $\bar{\ell}_j$  necessarily follows  $\bar{\ell}_n$  in each element of  $[\mathcal{L}]$ , contradicting the supposition that  $n \in \mathcal{A}$ .
4. Building on item 3 above, both endpoints of the  $j$ th arc in the half-plane diagram for  $[\mathcal{L}]$  with  $j \neq m, n$  reside in only one of the following four intervals:  $(x_{i_{2m}}, x_{i_{2m-1}})$ ,  $(x_{i_{2m-1}}, x_{i_{2n-1}})$ ,  $(x_{i_{2n-1}}, x_{i_{2n}})$ , or  $(x_{i_{2n}}, x_{i_{2m}})$ . Indeed, were the endpoints to reside in different intervals, then because the  $j$ th arc cannot cross the  $m$ th arc or the  $n$ th arc, one endpoint must be in  $(x_{i_{2m-1}}, x_{i_{2n-1}})$  while the other must be in  $(x_{i_{2n}}, x_{i_{2m}})$ . But this would contradict item 2 above.
  5. None of the coordinates of  $\mathbf{x} \in \Omega_0$  that are not involved in  $[\mathcal{L}]$  can reside between the endpoints of the  $j$ th arc with  $j \neq m$ . For if one such coordinate did reside there, then  $\ell_j = \underline{\ell}_j$ , and  $\underline{\ell}_j$  necessarily follows  $\underline{\ell}_m$  in all elements of  $[\mathcal{L}]$ , contradicting the supposition that  $m \in \mathcal{A}$ .

(Figure 10 illustrates an example half-plane diagram of an  $[\mathcal{L}]$  with  $M < N$  limits that satisfies the criteria listed in items 1–5 above.) Now, items 4–5 imply the existence of an allowable sequence  $\mathcal{L}_{\bar{m}, \bar{n}}$  of the  $M - 2$  limits of  $\{\ell_j\}_{j \neq m, n}$  that contract away all but the  $m$ th and  $n$ th arcs in the half-plane diagram for  $\mathcal{L}$ , and lemma 10 implies that  $\mathcal{L}_{\bar{m}, \bar{n}} F \in \mathcal{S}_{N-M+2}$ . Having proven this lemma for the case  $M = 2$  earlier, we immediately have from (91) that  $\underline{\ell}_m \ell_n \mathcal{L}_{\bar{m}, \bar{n}} F = \bar{\ell}_n \underline{\ell}_m \mathcal{L}_{\bar{m}, \bar{n}} F$ . But also,  $\bar{\ell}_n \mathcal{L}_{\bar{m}, \bar{n}} \in [\mathcal{L}_{\bar{m}}]$  and  $\underline{\ell}_m \mathcal{L}_{\bar{m}, \bar{n}} \in [\mathcal{L}_{\bar{n}}]$ , so this equality and the induction hypothesis imply that  $\underline{\ell}_m [\mathcal{L}_{\bar{m}}] F = \bar{\ell}_n [\mathcal{L}_{\bar{n}}] F$  for  $m, n \in \mathcal{A}$ . Thus,  $[\mathcal{L}] F$  is well-defined when  $M < N$ . The proof for the case where the arcs for  $\ell_m, \ell_n$  are un-nested proceeds similarly.

If  $N = M$  and the  $m$ th arc nests the  $n$ th arc, then  $[\mathcal{L}]$  must satisfy conditions 1–2 still, but it may violate conditions 3–4. (Item 5 is trivially true.) If  $[\mathcal{L}]$  satisfies all of conditions 1–5, then the argument of the previous paragraph proves that  $[\mathcal{L}] F$  is well-defined when  $M = N$ . If  $[\mathcal{L}]$  does not satisfy conditions 3–4, then we let  $j_1, \dots, j_k$  label the  $k$  arcs from outermost to innermost (not including the  $m$ th or  $n$ th arc). Now there are two facts to note.

I If  $M = N$ , then it is easy to see that  $\mathcal{A} = \{1, \dots, N\}$  (115).

II For all  $j \in \{1, \dots, N\}$ ,  $\bar{\ell}_j [\mathcal{L}_{j_1}] F = \underline{\ell}_j [\mathcal{L}_{j_1}] F$ . This follows immediately from the fact that  $[\mathcal{L}_{j_1}] F \in \mathcal{S}_1$  and (16).

Item I above and the argument of the previous paragraph prove that  $\underline{\ell}_m [\mathcal{L}_{\bar{m}}] F = \bar{\ell}_{j_1} [\mathcal{L}_{j_1}] F$ , then  $\underline{\ell}_{j_1} [\mathcal{L}_{j_1}] F = \bar{\ell}_{j_2} [\mathcal{L}_{j_2}] F$ , etc., until we reach  $\underline{\ell}_{j_{k-1}} [\mathcal{L}_{j_{k-1}}] F = \bar{\ell}_{j_k} [\mathcal{L}_{j_k}] F$ , and finally  $\underline{\ell}_{j_k} [\mathcal{L}_{j_k}] F = \bar{\ell}_n [\mathcal{L}_{\bar{n}}] F$ . Next, we can use item II to find

$$\underline{\ell}_m [\mathcal{L}_{\bar{m}}] F = \underline{\ell}_{j_1} [\mathcal{L}_{j_1}] F = \underline{\ell}_{j_2} [\mathcal{L}_{j_2}] F = \dots = \underline{\ell}_{j_k} [\mathcal{L}_{j_k}] F = \bar{\ell}_n [\mathcal{L}_{\bar{n}}] F, \quad (116)$$

so we conclude that  $[\mathcal{L}]F$  is well-defined when  $M = N$ . The proof of the case for which the  $m$ th and  $n$ th arcs are un-nested proceeds similarly.  $\square$

#### IV. AN UPPER BOUND FOR $\dim \mathcal{S}_N$

In this section we find an upper-bound for the rank of  $\mathcal{S}_N$ , namely  $\dim \mathcal{S}_N \leq C_N$ . First, we introduce some terminology motivated by CFT. In CFT, if each endpoint of an interval  $(x_i, x_{i+1})$  hosts a one-leg boundary operator, then upon sending  $x_{i+1} \rightarrow x_i$ , these operators fuse into a combination of an identity operator and a two-leg operator. If only the identity (resp. two-leg) channel appears in the OPE, then we call  $(x_i, x_{i+1})$  an *identity interval* (resp. a *two-leg interval*), and the correlation function with these one-leg boundary operators exhibits this OPE by admitting a Frobenius series expansion in  $x_{i+1}$  centered on  $x_i$  and with indicial power  $p_1 = -2\theta_1 + \theta_0 = 1 - 6/\kappa$  (resp.  $p_2 = -2\theta_1 - \theta_2 = 2/\kappa$ ) (29). If both of these channels appear in the OPE, then we call  $(x_i, x_{i+1})$  a *mixed interval*, and the correlation function equals a linear combination of these two Frobenius series. (The powers of these two expansions differ by an integer only when  $\kappa = 8/m$  for some  $m \in \mathbb{Z}^+$ , and in these cases, the two series coalesce into one. This invites the consideration of logarithmic CFT [44, 45]) Lemma 4 extends these notions to all elements of  $\mathcal{S}_N$  (should any not have interpretations as correlation functions of one-leg boundary operators), motivating the following definition.

**Definition 13.** Suppose that  $F \in \mathcal{S}_N$  and let  $i \in \{1, \dots, 2N-1\}$ . Interpreting  $\pi_{i+1}(\Omega_0)$  to be the part of the boundary of  $\Omega_0$  whose points have only their  $i$ th and  $(i+1)$ th coordinates equal, define

$$H : \Omega_0 \cup \pi_{i+1}(\Omega_0) \rightarrow \mathbb{R}, \quad H(\mathbf{x}) := (x_{i+1} - x_i)^{6/\kappa-1} F(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Omega_0, \quad (117)$$

and with the formula for  $H$  on  $\Omega_0$  continuously extended to  $\pi_{i+1}(\Omega_0)$ , and define

$$F_0 : \pi_{i+1}(\Omega_0) \rightarrow \mathbb{R}, \quad F_0(\pi_{i+1}(\mathbf{x})) := \lim_{x_{i+1} \rightarrow x_i} (x_{i+1} - x_i)^{6/\kappa-1} F(\mathbf{x}). \quad (118)$$

(Definition 118 is virtually identical to  $F_0$  defined in (76).) We call the open interval  $(x_i, x_{i+1})$

- a *two-leg interval* if  $F_0 = 0$  for all  $\kappa \in (0, 8)$ ,
- an *identity interval* if  $F_0 \neq 0$  for some  $\kappa \in (0, 8)$  and  $H$  is analytic at every point in  $\pi_{i+1}(\Omega_0)$  for all  $\kappa \in (0, 8)$ ,
- a *mixed interval* if  $F_0 \neq 0$  for some  $\kappa \in (0, 8)$  and  $H$  is not analytic at some point in  $\pi_{i+1}(\Omega_0)$  for some  $\kappa \in (0, 8)$ .

Let  $x'_i = f(x_i)$  where  $f$  is the Möbius transformation defined in (81), and let  $\mathbf{x}' = (x'_1, \dots, x'_{2N})$ . Then the interval  $(x_{2N}, x_1)$  is defined to be the same type as its image interval  $(x'_{2N}, x'_1)$  with respect to the function  $\hat{F}(\mathbf{x}')$  of (14) restricted to  $f(\Omega_0) := \{\mathbf{x}' \in \Omega \mid \mathbf{x} \in \Omega_0\}$ . (This restricted function of  $\mathbf{x}'$  is an element of  $\mathcal{S}_N$ .)

The Green function (38) used in the proof of lemma 3 gives the power law for  $F$  when  $(x_i, x_{i+1})$  is a two-leg interval. This is simply the two-leg power  $p_2 = -2\theta_1 + \theta_2 = 2/\kappa$  of (29), as we expect. We will prove this claim in appendix B as part of a suggested approach to proving the following conjecture.

**Conjecture 14.** Suppose that  $F \in \mathcal{S}_N$  and  $\kappa \in (0, 8)$ . If each of  $(x_1, x_2)$ ,  $(x_2, x_3)$ ,  $\dots$ ,  $(x_{2N-2}, x_{2N-1})$ ,  $(x_{2N-1}, x_{2N})$  is a two-leg interval, then  $F$  is zero.

The physical motivation for this conjecture is as follows. We recall from section IB that boundary arcs anchored to the endpoints of a two-leg interval are conditioned to not join in the long-time limit of multiple-SLE $_{\kappa}$ . If all of the intervals  $(x_1, x_2)$ ,  $(x_2, x_3)$ ,  $(x_3, x_4)$ ,  $\dots$ ,  $(x_{2N-1}, x_{2N})$  are two-leg intervals, then there cannot be an interval among these whose endpoints are joined by a single boundary arc. But, topological considerations show that no such boundary arc connectivity exists, thus implying conjecture 14.

It is interesting to note that conjecture 14 is not true if we omit the Ward identities (11) and only consider the  $2N$  null-state PDEs (10). Then (as long as the definition of a “two-leg interval” can be adapted for a solution of this smaller system), we have the following counterexample:

$$F(\mathbf{x}) = \prod_{i < j}^{2N} (x_j - x_i)^{2/\kappa}. \quad (119)$$

This function, for which each of the intervals  $(x_1, x_2)$ ,  $(x_2, x_3)$ ,  $\dots$ ,  $(x_{2N-2}, x_{2N-1})$ ,  $(x_{2N-1}, x_{2N})$  are two-leg intervals, satisfies the  $2N$  null-state PDEs (10) and the first Ward identity of (11) (counting from the left), but it does not satisfy the other Ward identities. Thus, any proof of conjecture 14 must employ the second and third Ward identities.

The weak maximum principle might seem to lend a more straightforward proof of conjecture 14 if  $\kappa \in (0, 6]$ , but this approach encounters technical difficulties. Such a proof might proceed as follows. We fix  $x_1 = a$  and  $x_{2N} = b$  with  $a < b$ , let  $F \in \mathcal{S}_N$ , define

$$\Omega_{a,b} := \{(x_2, \dots, x_{2N-1}) \in \mathbb{R}^{2N-2} \mid a < x_2 < \dots < x_{2N-1} < b\}, \quad (120)$$

and let  $F_R : \Omega_{a,b} \rightarrow \mathbb{R}$  be given by  $F_R(x_2, \dots, x_{2N-1}) = F(a, x_2, \dots, x_{2N-1}, b)$ . After using the first two Ward identities of (11) (counting from the left) to eliminate all derivatives in  $x_1$  and  $x_{2N}$ , the null-state PDE centered on  $x_j$  with  $j \in \{2, \dots, 2N-1\}$  becomes

$$\left[ \frac{\kappa}{4} \partial_j^2 + \sum_{k \neq j, 1, 2N} \left( \frac{\partial_k}{x_k - x_j} - \frac{(6 - \kappa)/2\kappa}{(x_k - x_j)^2} \right) - \sum_{k \neq 1, 2N} \frac{(x_k - a) \partial_k}{(b - x_j)(x_j - a)} \right. \\ \left. + \sum_{k \neq 1, 2N} \frac{\partial_k}{x_j - a} - \frac{N(6/\kappa - 1)}{(b - x_j)(x_j - a)} - \frac{(6 - \kappa)/2\kappa}{(x_j - a)^2} - \frac{(6 - \kappa)/2\kappa}{(b - x_j)^2} \right] F_R(x_2, \dots, x_{2N-1}) = 0. \quad (121)$$

If we sum (121) over  $j \in \{2, \dots, 2N-1\}$ , then we find a strictly elliptic PDE with a nonpositive constant term for  $\kappa \in (0, 6]$ . By hypothesis, all intervals of  $F_R$  are two-leg intervals. So according to lemmas 4 and 18,  $F_R$  continuously extends to and equals zero on  $\partial\Omega_{a,b} \setminus \pi_{1,2N}(E)$ , where  $E$  is the set of points  $(a, x_2, \dots, x_{2N-1}, b) \in \partial\Omega_0$  with three or more coordinates equal. If we show that  $F_R$  continuously extends to and equals zero on  $\pi_{1,2N}(E)$  as well, then we can invoke the weak maximum principle to prove conjecture 14 for  $\kappa \in (0, 6]$ . However, we have not found a way to show this, nor have we found bounds on the growth of  $F_R$  near  $\pi_{1,2N}(E)$  for which we could use a Phragmén-Lindelöf maximum principle to skirt this issue. Furthermore, this principle would be difficult to apply because the coefficients of (121) are not bounded as the points in  $\pi_{1,2N}(E)$  are approached. For this reason, it seems that a different method is needed, and one possibility is described in appendix B.

Supposing that conjecture 14 is true, the proof that  $\dim \mathcal{S}_N \leq C_N$  follows immediately.

**Lemma 15.** *Suppose that  $F \in \mathcal{S}_N$  and  $\kappa \in (0, 8)$ , and let  $v : \mathcal{S}_N \rightarrow \mathbb{R}^{C_N}$  be the map with the  $k$ th coordinate of  $v(F)$  equaling  $[\mathcal{L}_k]F$  for  $[\mathcal{L}_k] \in \mathcal{B}_N^*$ . If conjecture 14 is true, then  $v$  is a linear injection, and  $\dim \mathcal{S}_N \leq C_N$ .*

*Proof.* The map  $v$  is clearly linear. To show that it is injective, we argue that its kernel is trivial. Suppose that  $F$  is not zero. We construct an allowable sequence of limits  $\mathcal{L}$  such that  $\mathcal{L}F \neq 0$  as follows. Assuming conjecture 14,  $F$  has at least one mixed interval or identity interval  $(x_{i_1}, x_{i_2}) = (x_i, x_{i+1})$ . We let  $\bar{\ell}_1$  collapse this interval. Then according to lemma 5,  $\bar{\ell}_1 F \in \mathcal{S}_{N-1}$ . Also,  $\bar{\ell}_1 F$  is not zero because  $(x_{i_1}, x_{i_2})$  is not a two-leg interval, so by conjecture 14, it must have at least one identity interval or mixed interval  $(x_{i_3}, x_{i_4})$ . Repeating this process  $N-1$  times leaves us with a nonzero number  $\mathcal{L}F := \mathcal{L} = \ell_N \ell_{N-1} \dots \ell_2 \bar{\ell}_1 F$ . Thus,  $v(F) \neq 0$ , and  $\ker v = \{0\}$ . Finally, the dimension theorem of linear algebra then implies that  $\dim \mathcal{S}_N \leq C_N$ .  $\square$

To finish, we state a corollary that justifies our exclusive consideration of the system of PDEs (10–11) with an even number of independent variables. The proofs of lemmas 3–5, 10, 12, and 15 do not required this number to be even, so they remain true even when it is odd. We therefore let  $\mathcal{S}_{N+1/2}$  be defined exactly as  $\mathcal{S}_N$  for the system of PDEs (10–11) but with  $2N+1$  independent variables.

**Corollary 16.** *Suppose that  $F \in \mathcal{S}_{N+1/2}$  and  $\kappa \in (0, 8)$ . If conjecture 14 is true and  $\kappa \neq 6$ , then  $\mathcal{S}_{N+1/2} = \{0\}$ .*

*Proof.* We assume that  $F \in \mathcal{S}_{N+1/2}$  is not zero. Then as noted in the proof of lemma 15, there exists an allowable sequence of  $N$  limits  $\mathcal{L}$  such that  $F_0 := \mathcal{L}F$  is not zero. Lemma 5 shows that  $F_0$  is a function of only one  $x_j \in \{x_1, \dots, x_{2N+1}\}$  and satisfies the system

$$\frac{\kappa}{2} \partial_j^2 F_0(x_j) = 0, \quad \partial_j F_0(x_j) = 0, \quad [x_j \partial_j + (6 - \kappa)/2\kappa] F_0(x_j) = 0, \quad [x_j^2 \partial_j + (6 - \kappa)x_j/\kappa] F_0(x_j) = 0. \quad (122)$$

If  $\kappa \neq 6$ , then this system of equations implies that  $F_0$  is zero, a contradiction.  $\square$

We conjecture a similar corollary for the case  $\kappa = 6$ , and we suggest an incomplete proof of this conjecture, similar to that proposed for conjecture 14, at the end of appendix B.

**Conjecture 17.** *Suppose that  $F \in \mathcal{S}_{N+1/2}$  and  $\kappa = 6$ . Then  $\mathcal{S}_{N+1/2} = \mathbb{R}$ .*



## V. SUMMARY

In this article, we consider the solutions of the system of PDEs (10–11) (for  $\kappa \in (0, 8)$ , with  $\kappa$  the Schramm–Löwner Evolution (SLE $_{\kappa}$ ) parameter). These PDEs are respectively the conformal field theory (CFT) null-state conditions and Ward identities for a  $2N$ -point correlation function of one-leg boundary operators (21). These correlation functions may also be regarded as partition functions summing over side-alternating free/fixed boundary condition events. They are central to the characterization of a statistical cluster or loop model such as percolation, or more generally the Potts models and  $O(n)$  models, at the statistical mechanical critical point in the continuum limit. These equations also govern multiple-SLE $_{\kappa}$  partition functions.

We consider solutions  $F \in \mathcal{S}_N$ , where  $\mathcal{S}_N$  is the solution space for the system of PDEs (10–11) (with power-law growth, see definition 1). We use techniques of analysis to establish various facts about these solutions. In section II we prove three lemmas showing that the boundary behavior of any solution  $F$  is consistent with the formalism of the CFT operator product expansion (OPE), which is generally assumed to be valid in the physics literature. In lemma 3, we prove that any solution  $F(\mathbf{x})$  is  $O((x_{i+1} - x_i)^{1-6/\kappa})$  when (only) two adjacent coordinates  $x_i$  and  $x_{i+1}$  of  $\mathbf{x}$  are brought together. Lemmas 4 and 5 strengthen this result by proving, respectively, that the limit of  $(x_{i+1} - x_i)^{6/\kappa-1}F(\mathbf{x})$  as  $x_{i+1} \rightarrow x_i$  exists and is an element of  $\mathcal{S}_{N-1}$ . Next, in sections III and IV, we prove that  $\dim \mathcal{S}_N \leq C_N$  if conjecture 14 is true, where  $C_N$  is the  $N$ th Catalan number (4). We do this by explicitly constructing  $C_N$  elements of the dual space  $\mathcal{S}_N^*$  (definition 9) and using them as components of a linear map  $v$  that sends  $\mathcal{S}_N$  into  $\mathbb{R}^{C_N}$  and is injective if conjecture 14 is true (lemma 15). We investigate a possible proof of this conjecture in appendix B.

In the sequel to this article [14], we will prove that if  $\kappa \in (0, 8)$  and conjecture 14 is true, then the equality  $\dim \mathcal{S}_N = C_N$  holds and all elements of  $\mathcal{S}_N$  can be constructed using the CFT Coulomb gas (contour integral) formalism. In two other articles, we will establish a connection between degeneracies of the solution space  $\mathcal{S}_N$  for certain “exceptional speeds”  $\kappa$  and CFT minimal models [41], and apply our results to statistical mechanical systems at criticality, giving a general crossing formula for polygons, and paying special attention to crossings in rectangles, hexagons, and octagons [15].

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### Appendix A: The conformal field theory perspective

This appendix is included to help clarify the connections of the system of PDEs (10–11), which we have examined from a purely mathematical point of view, with aspects of CFT that we have mentioned throughout this article. Although the system of PDEs (10–11), to our knowledge, first appeared in CFT, that discipline is not necessary to understand our mathematical results about it. However, to understand our results from the CFT point of view is still useful and interesting. For this reason, we summarize here the main points of the application of CFT to study critical lattice models in simply-connected domains with a boundary that we have used, keeping in mind that intuitive notions will prevail over rigorous analysis. More on this subject can be found in [1–7, 46, 47].

In this appendix, we work with the continuum limit of the critical  $Q$ -state Potts model [18] in the upper half-plane. To begin, we turn our attention to conformally invariant boundary conditions (BC) for this system, of which there are several [7, 48, 49]. In the *fixed*, or *wired*, BC, we assign all boundary sites to the same state, and we denote by “ $a$ ” the fixed BC with all boundary sites in state  $a$ . Also, in the *free* BC, we do not condition the boundary sites, and we denote this BC by “ $f$ .” And also, we can condition the boundary sites to exhibit any but one of the  $Q$  possible states with uniform probability  $(Q - 1)^{-1}$ , and we denote by “ $\not{a}$ ” the BC with the state  $a$  excluded. The random cluster representation of the  $Q$ -state Potts model [18] also has conformally invariant boundary conditions. In the *fixed*, or *wired*, BC, we activate all bonds between boundary lattice sites, and we denote by “ $a$ ” the fixed BC with all bonds in state  $a$ . Also, in the *free* BC, we do not condition bonds between boundary sites, and we denote this BC by “ $f$ .”

We can condition the BC to change at (or near) a point  $x_0 \in \mathbb{R}$  by inserting a *boundary-condition-changing (BCC) operator*  $\phi(x_0)$ . (In CFT, the term “operator” is used to refer to random objects that may or may not have operator characteristics in a given physical system. They are perhaps best envisioned as random fields.) BCC operators are examples of *boundary operators*, or primary CFT operator with the system boundary as their domains. Under a

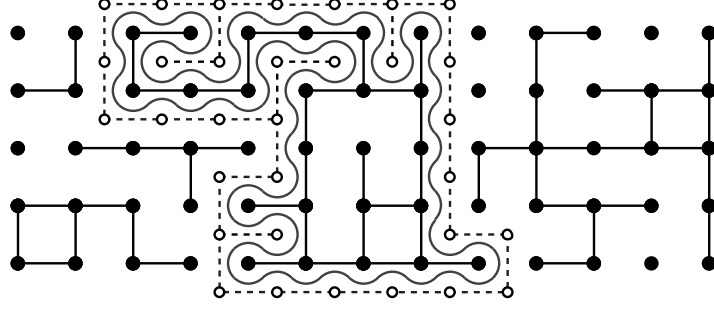


FIG. 11: The winding curve is a boundary loop surrounding a boundary FK cluster. If all sites inside (resp. outside) of this loop are (resp. are not) in spin state  $a$ , then the loop of activated dual bonds (dashed) is a boundary loop surrounding a boundary spin cluster.

conformal bijection  $f$  sending the upper half-plane onto another simply connected planar region, a primary operator transforms according to

$$\phi_h(x_0) \mapsto \phi'_h(f(x_0)) = |\partial f(x_0)|^{-h} \phi_h(x_0), \quad (\text{A1})$$

where  $h$  is called the *conformal weight* of  $\phi$ . *Kac operators*  $\phi_{r,s}$  are (boundary) primary operators whose conformal weight equals a *Kac weight*. These are the particular conformal weights

$$h_{r,s} = \frac{1}{16\kappa} \begin{cases} (\kappa r - 4s)^2 - (\kappa - 4)^2, & \kappa > 4 \\ (\kappa s - 4r)^2 - (\kappa - 4)^2, & \kappa \leq 4 \end{cases}, \quad r, s \in \mathbb{Z}^+. \quad (\text{A2})$$

In our application, Kac operators are of interest because many of them (and some with conformal weights (A2) with  $r$  and  $s$  equaling zero or a half-integer) can be used to induce certain BCCs. When primary operators appear in a correlation function, they give rise to PDEs known as null-state equations that govern the correlation function [2–4] in which they appear. For instance, in the random cluster model, a BCC at  $x_0$  from the  $a$  BC for  $x < x_0$  (resp.  $x > x_0$ ) to the  $f$  BC for  $x > x_0$  (resp.  $x < x_0$ ) is induced by the  $(1, 2)$  Kac operator, which we denote by  $\phi_{1,2}^{af}(x_0)$  (resp.  $\phi_{1,2}^{fa}(x_0)$ ) [7]. And in the  $Q$ -state Potts model, the BCC from the  $a$  BC for  $x < x_0$  (resp.  $x > x_0$ ) to the  $\emptyset$  BC for  $x > x_0$  (resp.  $x < x_0$ ) is induced by the  $(2, 1)$  Kac operator, which we denote by  $\phi_{2,1}^{a\emptyset}(x_0)$  (resp.  $\phi_{2,1}^{\emptyset a}(x_0)$ ) [7]. For later convenience, we will refer to either of these BCCs at  $x_0$  as *fixed-to-free* or *free-to-fixed*, even though this description is not completely accurate for  $\phi_{2,1}(x_0)$ .

The fixed-to-free or free-to-fixed BCC operator  $\phi_{1,2}(x_0)$  (resp.  $\phi_{2,1}(x_0)$ ) bears a second interpretation. As this BCC operator conditions an FK-cluster (resp. a spin-cluster) to anchor to the fixed side of  $x_0$ , it also conditions the interface, or boundary arc, between that cluster and the rest of the system to anchor to  $x_0$  [5]. For this reason, we also call this BCC operator a *one-leg boundary operator*. In the introduction, we stated that the boundary arc fluctuates in the system domain with the law of multiple-SLE $_{\kappa}$ . In particular, the speed  $\kappa$  that models FK-cluster interfaces in the random cluster model with  $Q$  states is related to  $Q$  through [34]

$$Q = 4 \cos^2(4\pi/\kappa), \quad \kappa \in (4, 8). \quad (\text{A3})$$

Because the  $Q$ -state Potts model can be mapped onto a random-cluster model with  $Q$  states [18], it corresponds to a CFT with central charge  $c(\kappa)$  (8) and  $\kappa$  given by (A3). Hence, if  $\hat{\kappa}$  is the speed that generates spin-cluster interfaces in the  $Q$ -state Potts model, then we need  $c(\kappa) = c(\hat{\kappa})$ . This implies the dual relation  $\hat{\kappa} = 16/\kappa$  [34]. We note that speeds  $\kappa$  generating FK-cluster interfaces reside in the dense phase  $(4, \infty)$  of SLE $_{\kappa}$ , while speeds  $\hat{\kappa}$  that generate spin-cluster interfaces reside in the dilute phase  $(0, 4]$ . This observation inspires us to collect  $\phi_{1,2}(x_0)$  and  $\phi_{2,1}(x_0)$  together with the notation

$$\psi_1(x_0) := \begin{cases} \phi_{1,2}(x_0), & \kappa > 4 \\ \phi_{2,1}(x_0), & \kappa \leq 4 \end{cases}. \quad (\text{A4})$$

After we declare it to be either a fixed-to-free or a free-to-fixed BCC operator,  $\psi_1(x_0)$  sums over all possible BCCs at  $x_0$  that are consistent with the chosen scenario. The collection of possible BCCs depends on the model under consideration. For example, in the  $Q$ -state Potts model, there are  $Q$  spin types for  $Q$  different fixed BCs and thus

$Q$  different free-to-fixed or fixed-to-free BCCs at  $x_0$ . We note from (A2) that its conformal weight, called the *one-leg boundary weight*, is (9)

$$\theta_1 := \begin{cases} h_{1,2}, & \kappa > 4 \\ h_{2,1}, & \kappa \leq 4 \end{cases} = \frac{6 - \kappa}{2\kappa}, \quad (\text{A5})$$

which, incidentally, is a continuous function at the transition  $\kappa = 4$  from the dilute phase to the dense phase.

Now we return our attention to a lattice model in a  $2N$ -sided polygon (conformally mapped onto the upper half-plane with vertices sent to  $x_1 < x_2 < \dots < x_{2N-1} < x_{2N}$ ). We let  $Z_\varsigma$  be the partition function summing over a specified free/fixed side-alternating BC (FFBC) event  $\varsigma$ . This is an event in which the BC alternates from free on  $(x_1, x_2)$  to fixed on  $(x_2, x_3)$  to free on  $(x_3, x_4)$ , etc., up to fixed on  $(x_{2N}, x_1)$ , or vice versa. At the critical point and in the continuum-limit, the ratio of  $Z_\varsigma$  to the free partition function  $Z_f$  summing over all system samples is supposed to behave as

$$Z_\varsigma/Z_f \underset{\epsilon_j \rightarrow 0}{\sim} C_1^{2N} \epsilon_1^{\theta_1} \dots \epsilon_{2N}^{\theta_1} \langle \psi_1(x_1) \dots \psi_1(x_{2N}) \rangle^\varsigma, \quad (\text{A6})$$

where  $\psi_1(x_i)$  implements the BCC within distance  $\epsilon_i$  from  $x_i$ , and where  $C_1$  is a non-universal constant associated with  $\psi_1$  and depending on the microscopic details of the discrete system that underlies the continuum limit. We interpret (A6) to say that each one-leg boundary operator induces the BCC near its location [1, 7]. According to the CFT null-state condition [2–4], the correlation function in (A6) satisfies the system of PDEs (10–11) central to this article. We interpret it as the  $\text{SLE}_\kappa$  partition function for a system conditioned on the FFBC event  $\varsigma$ . Determining which  $\text{SLE}_\kappa$  partition functions condition on which FFBC events will be the subject of [15].

In CFT, primary boundary operators exhibit an algebraic property called *fusion*. In short, two primary operators  $\phi(x_1)$  and  $\phi(x_2)$  whose respective points  $x_1$  and  $x_2$  approach each other may be replaced by a sum over other primary operators (and their “descendant” terms) within a correlation function. Because this replacement is useful only when  $x_1$  and  $x_2$  are very close, we write it as

$$\phi_1(x_1)\phi_2(x_2) \underset{x_2 \rightarrow x_1}{\sim} \sum_h C_{12}^h |x_2 - x_1|^{-h_1 - h_2 + h} \phi_h(x_1) + \dots \quad (\text{A7})$$

with  $h_1, h_2$ , and  $h$  the conformal weights of the primary operators  $\phi_1, \phi_2$  and  $\phi_h$ . The sum on the right side of (A7), called an *operator product expansion* (OPE), is the product of the fusion of  $\phi_1$  with  $\phi_2$  appearing on the left side of (A7), and each term in the sum is called a *fusion channel*. If the primary operators appearing on this left side are one-leg boundary operators, then replacing the left side of (A7) by its right side in the correlation function of (A6) implies that the correlation function has a Frobenius series expansion (22–23) in  $x_2$  centered on  $x_1$ . Because this correlation function satisfies the system of PDEs (10–11), we are naturally led to ask which, if not all, solutions to this system have such Frobenius series expansions. We used this question to guide the analysis that preceded lemma 3 in section II.

When the two operators on the left side of (A7) are Kac operators (as is the case with one-leg boundary operators), the content of their OPE is strongly constrained. For example, the OPE of two  $\phi_{1,2}$  (resp.  $\phi_{2,1}$ ) Kac operators can

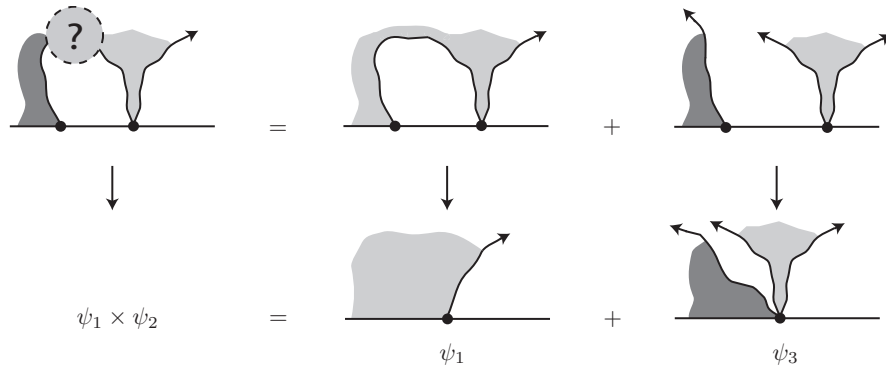


FIG. 12: The fusion rule  $\psi_1 \times \psi_2 = \psi_1 + \psi_3$ . (See (A14) for a definition of  $\psi_s$ .) In the dense phase, each  $s$ -leg operator sums over all possible states of all boundary clusters anchored to it. Our use of different shades of gray for the left versus right boundary cluster merely indicates this fact, and it does not indicate that these two clusters must exhibit different states. However, the  $\psi_1$  fusion channel is observed only for those samples in which both of these boundary clusters do exhibit the same state.

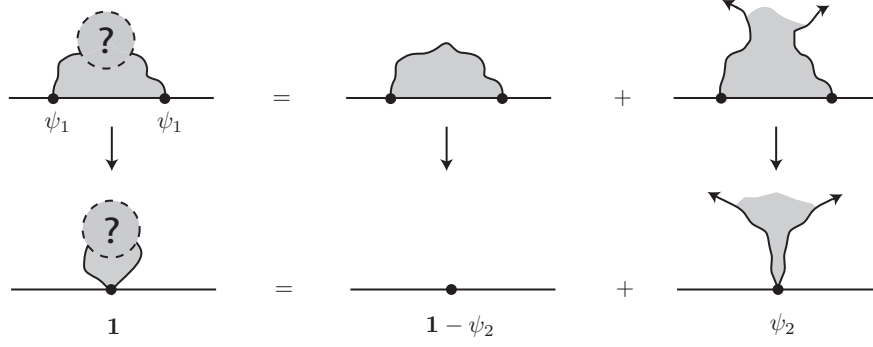


FIG. 13: The fusion rule  $\psi_1 \times \psi_1 = \mathbf{1} + \psi_2$ . The identity family does not condition the connectivity of the two boundary arcs with each other, and the two-leg family conditions the two boundary arcs to not connect with each other.

only contain a  $\phi_{1,1}$  (i.e., identity  $\mathbf{1}$ ) Kac operator and/or a  $\phi_{1,3}$  (resp.  $\phi_{3,1}$ ) Kac operator (and their descendants). We call this a *fusion rule*, and we write it as (after, as is customary in CFT, dropping the constant, the power of the difference  $x_2 - x_1$ , and the descendant terms for notational conciseness)

$$\begin{cases} \phi_{1,2}(x_1)\phi_{1,2}(x_2) \underset{x_2 \rightarrow x_1}{\sim} \mathbf{1} + \phi_{1,3}(x_1) \\ \phi_{2,1}(x_1)\phi_{2,1}(x_2) \underset{x_2 \rightarrow x_1}{\sim} \mathbf{1} + \phi_{3,1}(x_1) \end{cases} \quad (\text{A8})$$

General fusion rules of Kac operators are given in [2–4].

One-leg boundary operator fusion rules are intuitively understood in context with their interpretations as BCC operators. For example, if a BCC from state  $a$  to free takes place at  $x_1$  and a BCC from free to state  $b$  takes place at  $x_2$ , and we send  $x_2 \rightarrow x_1$ , then (A8) expressed in terms of the boundary operators that implement these BCCs reads

$$\phi_{1,2}^{af}(x_1)\phi_{1,2}^{fb}(x_2) \underset{x_2 \rightarrow x_1}{\sim} \delta_{ab}\mathbf{1}^{aa} + \phi_{1,3}^{ab}(x_1), \quad (\text{A9})$$

with the following physical interpretation. If  $a \neq b$ , then in this limit, the two BCCs join into one that takes us from fixed state  $a$  to fixed state  $b$  as we move rightward through  $x_1$ , and this fixed-to-fixed BCC is implemented by the boundary operator  $\phi_{1,3}^{ab}(x_1)$ . If  $a = b$ , then the lack of a BCC at  $x_1$  is captured by the identity term in the OPE. (An identity operator in a correlation function has no effect in the sense that is nonlocal and its removal does not alter the value of the correlation function.) The second term on the right side of (A9) includes all configurations with an infinitesimal free segment  $(x_1, x_2)$  that abuts a fixed segment on either end, so the FK boundary cluster anchored to the left fixed segment is disconnected from that anchored to the right fixed segment. These configurations are also included in the identity term in (A9). The order of the BCC may be switched so that we fuse  $\phi_{1,2}^{fa}(x_1)$  with  $\phi_{1,2}^{af}(x_2)$ . Here, we need both BCC operators to involve the same fixed BC for physical consistency, and we find a fusion rule similar to (A9):

$$\phi_{1,2}^{fa}(x_1)\phi_{1,2}^{af}(x_2) \underset{x_2 \rightarrow x_1}{\sim} \mathbf{1}^{ff} + \phi_{1,3}^{af}(x_1). \quad (\text{A10})$$

Now, the second boundary operator  $\phi_{1,3}^{af}(x_1)$  on the right side conditions the existence of an infinitesimal segment at  $x_1$  fixed to state  $a$ , and this forces a state  $a$  FK boundary cluster to anchor to this infinitesimal segment.

In either of these cases, we may continue this fusion process an arbitrary number of times to create an arbitrary number of BCCs proximal to  $x_1$ . This implies the general fusion rule

$$\phi_{1,2}(x_1)\phi_{1,s+1}(x_2) \underset{x_2 \rightarrow x_1}{\sim} \phi_{1,s}(x_1) + \phi_{1,s+2}(x_1), \quad (\text{A11})$$

with the superscripts indicating the BCCs suppressed. In this rule  $\phi_{1,s+1}(x_2)$  implements  $s$  distinct BCCs clumped very near  $x_2$ , and fusing  $\phi_{1,2}(x_1)$  with  $\phi_{1,s+1}(x_2)$  either annihilates one of these BCCs or adds a new one.

Now we turn our attention from FK-cluster perimeters to spin-cluster perimeters. If a BCC from state  $a$  to state  $\phi$  takes place at  $x_1$ , a BCC from state  $\phi$  to state  $a$  takes place at  $x_2$ , and we send  $x_2 \rightarrow x_1$ , then (A8) expressed in terms of the boundary operators that implement these BCCs reads

$$\phi_{2,1}^{a\phi}(x_1)\phi_{2,1}^{\phi a}(x_2) \underset{x_2 \rightarrow x_1}{\sim} \mathbf{1}^{aa} + \phi_{3,1}^{a\phi}(x_1). \quad (\text{A12})$$

The interpretation of (A12) is identical to that of (A9), except that  $\phi_{3,1}^{ada}(x_1)$  will now separate two disjoint boundary spin clusters with an infinitesimal free segment centered at  $x_1$ . A similar adaption of (A11) to this case gives the rule

$$\phi_{2,1}(x_1)\phi_{s+1,1}(x_2) \underset{x_2 \rightarrow x_1}{\sim} \phi_{s,1}(x_1) + \phi_{s+2,1}(x_1). \quad (\text{A13})$$

For the purposes of this article, it is more natural to interpret these fusion rules in terms of boundary arcs. Generalizing (A4), we define the  $s$ -leg boundary operator  $\psi_s$  with  $s$ -leg boundary weight  $\theta_s$ ,

$$\psi_s(x_0) = \begin{cases} \phi_{1,s+1}(x_0) & \kappa > 4 \\ \phi_{s+1,1}(x_0) & \kappa \leq 4 \end{cases}, \quad \theta_s := \begin{cases} h_{1,s+1} & \kappa > 4 \\ h_{s+1,1} & \kappa \leq 4 \end{cases} = \frac{s(2s+4-\kappa)}{2\kappa}, \quad (\text{A14})$$

so called because  $\psi_s$  conditions  $s$  distinct boundary arcs to emanate from the  $s$  BCCs tightly accumulated around  $x_0$ . These  $s$  BCCs alternate between fixed-to-free and free-to-fixed as we move rightward through  $x_0$ , starting with either of these BCCs furthest to the left. After we choose which of these two BCCs starts this alternating,  $\psi_s(x_0)$  sums over all possible BCCs at  $x_0$  that are consistent with this choice. Fusion rules (A11, A13) combine into

$$\psi_1(x_1)\psi_s(x_2) \underset{x_2 \rightarrow x_1}{\sim} \psi_{s-1}(x_1) + \psi_{s+1}(x_1), \quad s \in \mathbb{Z}^+, \quad (\text{A15})$$

which is interpreted as follows. If the boundary arc anchored to  $x_1$  connects with the leftmost of the boundary arcs anchored to  $x_2 > x_1$ , then this boundary arc contracts to a point almost surely as  $x_2 \rightarrow x_1$ , leaving  $s-1$  boundary arcs emanating from  $x_1$ . This explains the first term in (A15). On the other hand, if these two adjacent boundary arcs do not connect, then we find  $s+1$  distinct boundary arcs emanating from  $x_1$  after  $x_2 \rightarrow x_1$ . This explains the second term in (A15). (See figure 12 for the case  $s=2$ ). The case  $s=1$  is exceptional. On the right side of (A15), we have a zero-leg operator (i.e., the identity) and a two-leg operator. The two boundary arcs created by a pair of one-leg operators whose fusion contains only the identity (resp. two-leg) family are not (resp. are) conditioned to join and form one boundary arc (figure 13).

## Appendix B: A proposal for the proof of conjectures 14 and 17

In this appendix, we propose an approach for proving conjectures 14 and 17. A proof of the former is the only missing ingredient needed to prove  $\dim \mathcal{S}_N \leq C_N$  (lemma 15). Our method is motivated by considering the behavior of a solution  $F \in \mathcal{S}_N$  when one of its two-leg intervals is collapsed. The following lemma, motivated by the OPE in CFT, explains this behavior.

**Lemma 18.** *Suppose that  $F \in \mathcal{S}_N$  and  $\kappa \in (0, 8)$ . If, for some  $i \in \{1, \dots, 2N-1\}$ ,  $(x_i, x_{i+1})$  is a two-leg interval, then the limit*

$$F_2(\pi_{i+1}(\mathbf{x})) := \lim_{x_{i+1} \rightarrow x_i} (x_{i+1} - x_i)^{2/\kappa} F(\mathbf{x}) \quad (\text{B1})$$

*exists for all points  $\pi_{i+1}(\mathbf{x}) \in \pi_{i+1}(\Omega_0)$ . Furthermore, if  $F$  is not zero, then  $F_2$  is not zero. Finally, for each  $j \in \{1, \dots, i-1, i+2, \dots, 2N\}$ ,  $F_2$  satisfies the (modified) null-state PDE centered on  $x_j$ ,*

$$\left[ \frac{\kappa}{4} \partial_j^2 + \sum_{k \neq j, i, i+1}^{2N} \left( \frac{\partial_j}{x_k - x_j} - \frac{\theta_1}{(x_k - x_j)^2} \right) + \frac{\partial_i}{x_i - x_j} - \frac{\theta_2}{(x_i - x_j)^2} \right] F_2(\pi_{i+1}(\mathbf{x})) = 0, \quad (\text{B2})$$

*where  $\theta_1 := (6 - \kappa)/2\kappa$  and  $\theta_2 := 8/\kappa - 1$ , and  $F_2$  satisfies the (modified) Ward identities*

$$\begin{aligned} & \sum_{k \neq i+1}^{2N} \partial_k F_2(\pi_{i+1}(\mathbf{x})) = 0, \\ & \left[ \sum_{k \neq i, i+1}^{2N} (x_k \partial_k + \theta_1) + x_i \partial_i + \theta_2 \right] F_2(\pi_{i+1}(\mathbf{x})) = 0, \\ & \left[ \sum_{k \neq i, i+1}^{2N} (x_k^2 \partial_k + 2\theta_1 x_k) + x_i^2 \partial_i + 2\theta_2 x_i \right] F_2(\pi_{i+1}(\mathbf{x})) = 0. \end{aligned} \quad (\text{B3})$$

*Proof.* To begin, we let  $F$ ,  $H$ ,  $\xi$ ,  $x$ ,  $\mathcal{K}_x$ ,  $\delta$ , and  $b$  be defined as in the proof of lemma 3, and we use the integral equations (74–75) to prove that for any compact subset  $\mathcal{K} \subset \subset \pi_{i+1}(\Omega_0)$ ,  $\sup_{\mathcal{K}} |F(\xi; x, \delta)| = O(\delta^{2/\kappa})$  as  $\delta \downarrow 0$ . Taking the supremum of (75) over an open set  $\mathcal{U}_1$  with  $\mathcal{K}_x \subset \subset \mathcal{U}_1 \subset \subset \pi_{i,i+1}(\Omega_0)$  and applying the two-leg interval condition  $H(\xi; x, 0) = 0$  (definition 13) gives

$$\sup_{\mathcal{U}_1} |F(\xi; x, \delta)| \leq \left(\frac{\delta}{b}\right)^{2/\kappa} \sup_{\mathcal{U}_1} |F(\xi; x, b)| + \frac{4\delta^{2/\kappa}}{\kappa} \int_{\delta}^b \int_0^{\beta} \beta^{-8/\kappa} \eta^{6/\kappa-1} \sup_{\mathcal{U}_1} |\eta \mathcal{M}[F](\xi; x, \eta)| d\eta d\beta. \quad (\text{B4})$$

Lemma 3 proves that  $\eta^{6/\kappa-1}$  times the supremum in the integrand is bounded over  $0 < \eta < b$ . After using this fact to estimate the double-integral in (B4), we find that the supremum of the left side of (B4) is  $O(\delta^p)$  with  $p = \min\{2/\kappa, 2 - 6/\kappa\}$ . The Schauder estimate (50) further implies that for all  $j, k \in \{1, \dots, 2N - 4\}$ ,

$$\sup_{\mathcal{U}_2} |F(\xi; x, \delta)|, \quad \sup_{\mathcal{U}_2} |\partial_j F(\xi; x, \delta)|, \quad \sup_{\mathcal{U}_2} |\partial_j \partial_k F(\xi; x, \delta)|, \quad \sup_{\mathcal{U}_2} |\delta \mathcal{M}[F](\xi; x, \delta)| \quad (\text{B5})$$

are  $O(\delta^p)$  as  $\delta \downarrow 0$ , where  $\mathcal{U}_2$  is an open set with  $\mathcal{K}_x \subset \subset \mathcal{U}_2 \subset \subset \mathcal{U}_1$ . If  $p = 2/\kappa$ , then we are finished. Otherwise, if  $p = 2 - 6/\kappa$ , then we insert this result back into (75), and repeat these steps as many times as necessary (exactly as we did in the proof of lemma 3) to ultimately find that

$$\sup_{\mathcal{K}} |F(\xi; x, \delta)|, \quad \sup_{\mathcal{K}} |\partial_j F(\xi; x, \delta)|, \quad \sup_{\mathcal{K}} |\partial_j \partial_k F(\xi; x, \delta)|, \quad \sup_{\mathcal{K}} |\delta \mathcal{M}[F](\xi; x, \delta)| \quad (\text{B6})$$

are  $O(\delta^{2/\kappa})$  as  $\delta \downarrow 0$ .

Now we use this result to show that the limit (B1) exists and is approached uniformly over  $\mathcal{K}$ . The reasoning follows the proof of lemma 4. We let

$$K(\xi; x, \delta) := \delta^{-2/\kappa} F(\xi; x, \delta), \quad (\text{B7})$$

which is bounded over  $0 < \delta < b$ , and we show that its superior limit and inferior limit as  $\delta \downarrow 0$  are equal. In terms of  $K$ , (75) with the two-leg condition  $H(\xi; x, 0) = 0$  gives

$$K(\xi; x, \delta) = K(\xi; x, b) - \frac{4}{\kappa} \int_{\delta}^b \frac{1}{\beta} \int_0^{\beta} \left(\frac{\eta}{\beta}\right)^{8/\kappa-1} \eta \mathcal{M}[K](\xi; x, \eta) d\eta d\beta, \quad 0 < \delta < b. \quad (\text{B8})$$

Because the quantities of (B6) are  $O(\delta^{2/\kappa})$  as  $\delta \downarrow 0$ , the integrand of the  $\eta$  integral is bounded over  $0 < \eta < b$ . Moreover, the factor  $(\eta/\beta)^{8/\kappa-1}$  in the integrand is bounded by one because  $\kappa < 8$  and  $0 < \eta < \beta$ . Hence,

$$\sup_{0 < \delta < b} |K(\xi; x, \delta) - K(\xi; x, b)| \longrightarrow 0 \quad \text{as } b \downarrow 0. \quad (\text{B9})$$

This implies that the superior and inferior limits of  $K(\xi; x, \delta)$  as  $\delta \downarrow 0$  are equal, and therefore the limit  $F_2(\xi; x) := \lim_{\delta \downarrow 0} K(\xi; x, \delta)$ , or (B1), exists. After taking the supremum of (B8) over  $\mathcal{K}$ , setting  $\delta = 0$ , and then replacing  $b$  with  $\delta$ , we also find

$$\sup_{\mathcal{K}} |K(\xi; y, \delta) - F_2(\xi; y)| \leq \frac{4}{\kappa} \int_0^{\delta} \frac{1}{\beta} \int_0^{\beta} \sup_{\mathcal{K}} |\eta \mathcal{M}[K](\xi; x, \eta)| d\eta d\beta \quad (\text{B10})$$

$$\longrightarrow 0 \quad \text{as } \delta \downarrow 0. \quad (\text{B11})$$

The limit follows because the quantities in (B6) are  $O(\delta^{2/\kappa})$ , so the supremum on the right side of (B10) is bounded over  $0 < \eta < b$ . Hence, the limit  $F_2$  is approached uniformly over  $\mathcal{K}$ . We can show that the same is true of the first and second partial derivatives of  $K$  with respect to the coordinates of  $\xi$  by following the same procedure. The details are identical to those in the proof of lemma 4.

Next, we prove that if  $F_2$  is zero, then  $F$  is zero. From (B8), after setting  $\delta = 0$  and replacing  $b$  with  $\delta$ , using that  $F_2 = 0$  and the estimate  $(\eta/\beta)^{8/\kappa-1} < 1$  for  $\kappa < 8$ , and taking the supremum over an open ball  $\mathcal{B}_1 \subset \subset \mathcal{K}_x$ , we find

$$\sup_{\mathcal{B}_1} |K(\xi; x, \delta)| \leq \frac{4}{\kappa} \int_0^{\delta} \frac{1}{\beta} \int_0^{\beta} \sup_{\mathcal{B}_1} |\eta \mathcal{M}[K](\xi; x, \eta)| d\eta d\beta. \quad (\text{B12})$$

The discussion below (B10) shows that the integrand of the first integral is bounded over  $0 < \eta < b$ . This fact and (51) give

$$\sup_{\mathcal{B}_1} |K(\xi; x, \delta)| \leq c_1(x)\delta, \quad d_1 \sup_{\mathcal{B}_2} |\delta \mathcal{M}[K](\xi; x, \delta)| \leq c_1(x)c_2(x)\delta, \quad (\text{B13})$$



for positive functions  $c_1(x)$  and  $c_2(x) := C'(x, R_1)$  (see (51)) and an open ball  $\mathcal{B}_2$  concentric with  $\mathcal{B}_1$  and with radius  $R_2 < R_1$ .

Next, we iterate the former estimate an infinite number of times. We let  $\dots \subset \mathcal{B}_{k+1} \subset \mathcal{B}_k \subset \dots \subset \mathcal{B}_2 \subset \mathcal{B}_1$  be an infinite sequence of concentric balls, with  $R_k$  the radius of  $\mathcal{B}_k$ , such that their intersection  $\bigcap_k \mathcal{B}_k$  is a common ball  $\mathcal{B}_\infty$  of radius  $R_\infty < R_k$  for all  $k \in \mathbb{Z}^+$ . We choose the radii of the balls such that

$$d_k := R_k - R_{k+1} = \frac{d_0}{(k+1)^{3/2}}, \quad d_0 := \frac{R_1 - R_\infty}{\zeta(3/2) - 1}, \quad (\text{B14})$$

with  $\zeta$  the Riemann zeta function. This choice satisfies the necessary condition  $\sum_{k=1}^\infty d_k = R_1 - R_\infty$ . After using (B13) to estimate the integral in (B12) with  $\mathcal{B}_2$  replacing  $\mathcal{B}_1$  and then using (51) again, we find

$$\sup_{\mathcal{B}_2} |K(\xi; x, \delta)| \leq \frac{4}{\kappa} c_1(x) c_2(x) \frac{\delta^2}{2^2 d_1}, \quad d_2 \sup_{\mathcal{B}_3} |\delta \mathcal{M}[K](\xi; x, \delta)| \leq \frac{4}{\kappa} c_1(x) c_2(x) c_3(x) \frac{\delta^2}{2^2 d_1}, \quad (\text{B15})$$

with  $c_3(x) := C'(x, R_2)$ . After we repeat this process another  $k-2$  times, we find

$$\sup_{\mathcal{B}_k} |K(\xi; x, \delta)| \leq \frac{\kappa}{4} c_1(x) c_2(x) \dots c_{k-1}(x) c_k(x) \frac{(4\delta/\kappa)^k}{(k!)^2 d_1 \dots d_{k-1}}, \quad k \in \mathbb{Z}^+, \quad (\text{B16})$$

with  $c_k(x) := C'(x, R_{k-1})$ . Because  $C'(x, R)$  is a continuous function of  $R$  on  $(0, R_1)$ , the sequence  $c_k(x)$  is pointwise bounded. Therefore, after substituting the formula for  $d_k$  (B14) into (B16) and recalling that  $\mathcal{B}_\infty \subset \mathcal{B}_k$  for all  $k \in \mathbb{Z}^+$ , we find

$$\sup_{\mathcal{B}_\infty} |K(\xi; x, \delta)| \leq \frac{\kappa d_0}{4\sqrt{k!}} \left( \frac{4}{\kappa d_0} \sup_{k \in \mathbb{Z}^+} c_k(x) \delta \right)^k, \quad k \in \mathbb{Z}^+ \quad (\text{B17})$$

$$\longrightarrow 0 \quad \text{as } k \rightarrow \infty. \quad (\text{B18})$$

Because  $\mathcal{B}_\infty$  is an arbitrary ball in  $\mathcal{K}_x$  and  $\mathcal{K}$  is an arbitrary subset of  $\Omega_0$ , it follows that  $K$ , and therefore  $F$  (B7), is zero.

To show that  $F_2(\xi; x)$  satisfies (B2–B3), we insert (B7) into the null-state PDEs centered on  $x_j$  with  $j \neq i, i+1$  and into the three Ward identities, multiply everything by  $\delta^{-2/\kappa}$ , and send  $\delta \downarrow 0$ . Because the limit of each term is approached uniformly over compact subsets of  $\pi_{i+1}(\Omega_0)$ , we may commute the limit with the derivatives appearing in these PDEs to find (B2–B3).  $\square$

In CFT parlance where we interpret  $F \in \mathcal{S}_N$  as a correlation function of  $2N$  one-leg boundary operators, saying that  $(x_i, x_{i+1})$  is a two-leg interval of  $F$  means that the OPE of the one-leg boundary operators  $\psi_1(x_i)$  and  $\psi_1(x_{i+1})$  contains only the two-leg fusion channel. Thus, it is natural to interpret  $F_2$  (B1) as, to within a factor, a  $(2N-1)$ -point correlation function with a two-leg boundary operator at  $x_i$  and a one-leg boundary operator at each of  $x_1, \dots, x_{i-1}, x_{i+2}, \dots, x_{2N}$ . This  $(2N-1)$ -point function is the leading term in the aforementioned OPE, which has the structure (A7)

$$\overbrace{\langle \psi_1(x_1) \dots \psi_1(x_{i-1}) \psi_1(x_i) \psi_1(x_{i+1}) \psi_1(x_{i+2}) \dots \psi_1(x_{2N}) \rangle}^F \underset{x_{i+1} \rightarrow x_i}{\sim} (x_{i+1} - x_i)^{-\theta_1 - \theta_1 + \theta_2} \underbrace{C_{11}^2 \langle \psi_1(x_1) \dots \psi_1(x_{i-1}) \psi_2(x_i) \psi_1(x_{i+2}) \dots \psi_1(x_{2N}) \rangle}_{F_2}, \quad (\text{B19})$$

with  $-2\theta_1 + \theta_2 = 2/\kappa$  and with  $\theta_s$  is given in (A14). CFT then asserts that  $F_2$  satisfies the null-state PDEs (B2) associated with the  $2N-2$  one-leg boundary operators on the right side of (B19). Because  $x_i$  now hosts a two-leg boundary operator whose conformal weight is  $\theta_2$  rather than  $\theta_1$ , the original null-state PDEs (10) centered on  $x_j$  with  $j \neq i, i+1$  are modified to (B2) to account for this change. (CFT also asserts that  $F_2$  satisfies another PDE not included among (B2–B3). This is the null-state PDE associated with the two-leg boundary operator  $\psi_2(x_i)$ , and it is given in [2, 50], but because we do need this PDE for our arguments below, we do not show it here.)

With this understanding, we can prove an adaptation of lemmas 3–5, and 18 to lemmas concerning the limiting behavior of  $F_2(\pi_{i+1}(\mathbf{x}))$  as  $x_{i+2} \rightarrow x_i$ . For example, by studying the null-state PDE (B2) with  $j = i+2$ , we can show that

$$F_2(\pi_{i+1}(\mathbf{x})) = O((x_{i+2} - x_i)^{-\theta_2}), \quad \text{as } x_{i+2} \rightarrow x_i. \quad (\text{B20})$$

The proof is identical to that of lemma 3. Furthermore, we can prove that the limit of  $(x_{i+2} - x_i)^{\theta_2} F_2(\pi_{i+1}(\mathbf{x}))$  as  $x_{i+2} \rightarrow x_i$  exists, and if it is zero, then the limit  $F_3(\pi_{i+1, i+2}(\mathbf{x}))$  of  $(x_{i+2} - x_i)^{\theta_1 + \theta_2 - \theta_3} F_2(\pi_{i+1}(\mathbf{x}))$  as  $x_{i+2} \rightarrow x_i$  exists, is not zero, and satisfies the modified null-state PDEs and Ward identities (B2-B3) with the index  $i + 2$  dropped and with  $\theta_2$  replaced by  $\theta_3$  (A14). The proofs of these two claims are respectively identical to the proofs of lemmas 4 and 18, and the latter situation is consistent with an OPE of  $\psi_2(x_i)$  with  $\psi_1(x_{i+2})$  that leads with the three-leg boundary operator:

$$\overbrace{C_{11}^2 \langle \psi_1(x_1) \dots \psi_1(x_{i-1}) \psi_2(x_i) \psi_1(x_{i+2}) \psi_1(x_{i+3}) \dots \psi_1(x_{2N}) \rangle}^{F_2} \underset{x_{i+2} \rightarrow x_i}{\sim} (x_{i+2} - x_i)^{-\theta_1 - \theta_2 + \theta_3} \underbrace{C_{11}^2 C_{21}^3 \langle \psi_1(x_1) \dots \psi_1(x_{i-1}) \psi_3(x_i) \psi_1(x_{i+3}) \dots \psi_1(x_{2N}) \rangle}_{F_3}. \quad (\text{B21})$$

After repeating this process another  $s - 3$  times, we arrive with the  $(2N - s + 1)$ -point function

$$F_s(\pi_{i+1, \dots, i+s-1}(\mathbf{x})) := \prod_{k=2}^s C_{k-1, 1}^k \langle \psi_1(x_1) \dots \psi_1(x_{i-1}) \psi_s(x_i) \psi_1(x_{i+s}) \dots \psi_1(x_{2N}) \rangle. \quad (\text{B22})$$

Here,  $C_{s,1}^{s+1}$  is the OPE coefficient of the fusion  $\psi_s \times \psi_1 = \psi_{s+1}$  [51]. The usual CFT null-state condition says that  $F_s$  must satisfy the system of  $2N - s$  null-state PDEs

$$\left[ \frac{\kappa}{4} \partial_j^2 + \sum_{k \neq j, i, \dots, i+s-1}^{2N} \left( \frac{\partial_j}{x_k - x_j} - \frac{\theta_1}{(x_k - x_j)^2} \right) + \frac{\partial_i}{x_i - x_j} - \frac{\theta_s}{(x_i - x_j)^2} \right] F_s(\pi_{i+1, \dots, i+s-1}(\mathbf{x})) = 0, \quad (\text{B23})$$

associated with the one-leg boundary operators  $\psi_1(x_j)$  of (B22) with  $j \in \{1, \dots, i-1, i+s, \dots, 2N\}$ , and the (modified) Ward identities

$$\begin{aligned} & \sum_{j \neq i+1, \dots, i+s-1}^{2N} \partial_j F_s(\pi_{i+1, \dots, i+s-1}(\mathbf{x})) = 0, \\ & \left[ \sum_{j \neq i+1, \dots, i+s-1}^{2N} (x_j \partial_j + \theta_1) + x_i \partial_i + \theta_s \right] F_s(\pi_{i+1, \dots, i+s-1}(\mathbf{x})) = 0, \\ & \left[ \sum_{j \neq i+1, \dots, i+s-1}^{2N} (x_j^2 \partial_j + 2\theta_1 x_j) + x_i^2 \partial_i + 2\theta_s x_i \right] F_s(\pi_{i+1, \dots, i+s-1}(\mathbf{x})) = 0. \end{aligned} \quad (\text{B24})$$

Here,  $\theta_s$  is the conformal weight of the  $s$ -leg operator (A14). This system of PDEs implies the growth condition

$$F_s(\pi_{i+1, \dots, i+s-1}(\mathbf{x})) = O((x_{i+s} - x_i)^{-\theta_1 - \theta_s + \theta_{s-1}}) \quad \text{as } x_{i+s} \rightarrow x_i. \quad (\text{B25})$$

The proof is again identical to the proof of lemma 3, with a few slight alterations as follows. After letting  $\delta := x_{i+s} - x_i$  and relabeling the variables  $\{x_j\}_{j \neq i, \dots, i+s}$  as  $\{\xi_1, \dots, \xi_{2N-s-1}\}$  in ascending order, we find that the differential operator  $\mathcal{L}$  of (36) is now

$$\mathcal{L}_s := \frac{\kappa}{4} \partial_\delta^2 + \frac{\partial_\delta}{\delta} - \frac{\theta_s}{\delta^2}, \quad (\text{B26})$$

while  $\mathcal{M}$  does not change (except that the number of coordinates for  $\boldsymbol{\xi}$  decreases to  $2N - s - 1$ ). The characteristic powers of  $\mathcal{L}_s$  are

$$p_1 = -\theta_1 - \theta_s + \theta_{s-1} = 1 - 2(s+2)/\kappa, \quad p_2 = -\theta_1 - \theta_s + \theta_{s+1} = 2s/\kappa, \quad (\text{B27})$$

and the Green function (38) is altered to

$$G_s(\delta, \eta) = \frac{4\eta}{\kappa(\theta_{s+1} - \theta_{s-1})} \Theta(\eta - \delta) \left[ \left( \frac{\delta}{\eta} \right)^{-\theta_1 - \theta_s + \theta_{s-1}} - \left( \frac{\delta}{\eta} \right)^{-\theta_1 - \theta_s + \theta_{s+1}} \right]. \quad (\text{B28})$$

Furthermore, we may also prove that the limit of  $(x_{i+s} - x_i)^{\theta_1 + \theta_s - \theta_{s-1}} F_s(\pi_{i, \dots, i+s-1}(\mathbf{x}))$  as  $x_{i+s} \rightarrow x_i$  exists, and if it is zero, then the limit  $F_{s+1}(\pi_{i, \dots, i+s}(\mathbf{x}))$  of  $(x_{i+s} - x_i)^{\theta_1 + \theta_s - \theta_{s+1}} F_s(\pi_{i, \dots, i+s-1}(\mathbf{x}))$  as  $x_{i+s} \rightarrow x_i$  exists, is not zero,

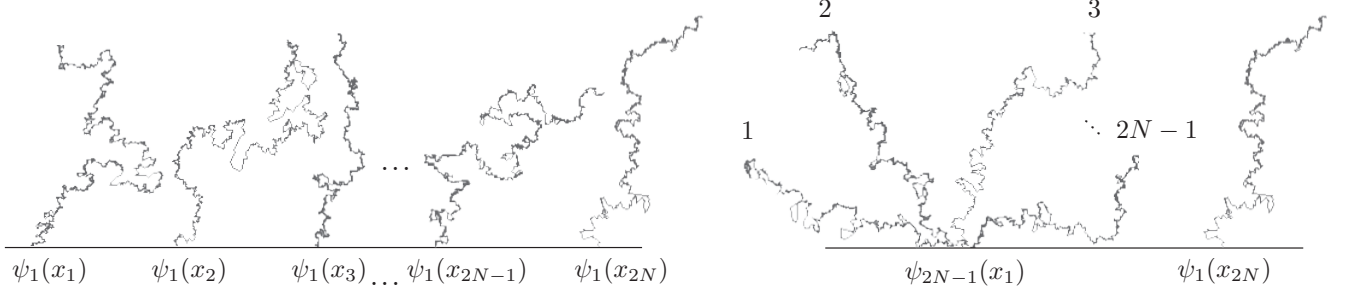


FIG. 14: The existence of an element of  $\mathcal{S}_N$  for which all intervals among  $(x_1, x_2), (x_2, x_3), \dots, (x_{2N-2}, x_{2N-1}), (x_{2N-1}, x_{2N})$  are two-leg intervals implies the existence of a nonzero two-point function of a  $(2N-1)$ -leg boundary operator and a one-leg boundary operator, an impossibility.

and satisfies the system (B23–B24) with  $s \mapsto s+1$ . The proof of these two claims are respectively identical to the proofs of lemmas 4 and 18, and the latter situation is consistent with an OPE of  $\psi_s(x_i)$  with  $\psi_1(x_{i+s})$  that leads with the  $(s+1)$ -leg boundary operator:

$$\overbrace{C_{11}^2 \dots C_{s-1,1}^s \langle \psi_1(x_1) \dots \psi_1(x_{i-1}) \psi_s(x_i) \psi_1(x_{i+s}) \psi_1(x_{i+s+1}) \dots \psi_1(x_{2N}) \rangle}^{F_s} \underset{x_{i+s} \rightarrow x_i}{\sim} (x_{i+s} - x_i)^{-\theta_1 - \theta_s + \theta_{s+1}} \underbrace{C_{11}^2 \dots C_{s+1}^{s+1} \langle \psi_1(x_1) \dots \psi_1(x_{i-1}) \psi_{s+1}(x_i) \psi_1(x_{i+s+1}) \dots \psi_1(x_{2N}) \rangle}_{F_{s+1}}. \quad (\text{B29})$$

The case  $s=1$  reproduces lemmas 3–5 and 18 with  $F_1 := F \in \mathcal{S}_N$ .

These observations serve as a significant part of the proof that  $\dim \mathcal{S}_N \leq C_N$  according to this reasoning. First, it is reasonable to suppose that which of the two power laws among (29) that  $F(\mathbf{x})$  exhibits as  $x_{i+1} \rightarrow x_{i+1}$  and as  $x_{i+2} \rightarrow x_{i+1}$  determines which of the two power laws among (B27) that  $F_2(\pi_{i+1}(\mathbf{x}))$  will exhibit as  $x_{i+2} \rightarrow x_i$ . For instance, we suppose that the adjacent intervals  $(x_i, x_{i+1})$ , and  $(x_{i+1}, x_{i+2})$  are two-leg intervals. In appendix A, we noted that in the multiple-SLE $_{\kappa}$  picture, the two endpoints of a boundary arc cannot also be the two endpoints of a two-leg interval. Therefore, no boundary arc can have both of its endpoints among  $\{x_i, x_{i+1}, x_{i+2}\}$ , so the limit  $x_{i+1} \rightarrow x_i$  followed by the limit  $x_{i+2} \rightarrow x_i$  produces a three-leg boundary operator at  $x_i$ . Consequently, it should be possible to prove that if  $(x_i, x_{i+1})$ , and  $(x_{i+1}, x_{i+2})$  are two-leg intervals of  $F \in \mathcal{S}_N$ , then  $F_2(\pi_{i+1}(\mathbf{x})) = O((x_{i+2} - x_i)^{-\theta_1 - \theta_2 + \theta_3})$  as  $x_{i+2} \rightarrow x_i$ .

Next, we suppose that each of  $(x_i, x_{i+1}), (x_{i+1}, x_{i+2}), (x_{i+2}, x_{i+3}), \dots, (x_{i+s-1}, x_{i+s})$  is a two-leg interval. Again, this implies that no boundary arc can have both of its endpoints among  $S = \{x_i, \dots, x_{i+s}\}$ , for otherwise, topological considerations show that another boundary arc must have its endpoints at some  $x_j, x_{j+1} \in S$ , contradicting that  $(x_j, x_{j+1})$  is a two-leg interval. Then the argument of the previous paragraph suggests that

$$F_s(\pi_{i+1, \dots, i+s-1}(\mathbf{x})) = O((x_{i+s} - x_i)^{-\theta_1 - \theta_s + \theta_{s+1}}) \quad \text{as } x_{i+s} \rightarrow x_i. \quad (\text{B30})$$

Again, it should be possible to prove (B30), and the proof will likely be a straightforward generalization of the proof for the case  $s=2$  discussed in the previous paragraph.

In order to complete the proof of conjecture 14, we only need to prove (B30) for  $i=1$  and  $s \in \{2, \dots, 2N-2\}$ . (The case  $s=1$  is proven in lemma 18.) Indeed, if  $F \in \mathcal{S}_N$  is not zero,  $(x_1, x_2), (x_2, x_3), \dots, (x_{2N-2}, x_{2N-1})$  are two-leg intervals, and (B30) is true for  $i=1$  and these values of  $s$ , then our previous arguments lead to the collection of functions  $F_2, \dots, F_{2N-1}$ , none of which are zero. In particular,  $F_{2N-1}$  is not zero and satisfies the three modified Ward identities (B3) with  $s=2N-1$  and  $i=1$ . This system is

$$[\partial_1 + \partial_{2N}] F_{2N-1}(x_1, x_{2N}) = 0, \quad (\text{B31})$$

$$[x_1 \partial_1 + x_{2N} \partial_{2N} + \theta_{2N-1} + \theta_1] F_{2N-1}(x_1, x_{2N}) = 0, \quad (\text{B32})$$

$$[x_1^2 \partial_1 + x_{2N}^2 \partial_{2N} + 2x_1 \theta_{2N-1} + 2x_{2N} \theta_1] F_{2N-1}(x_1, x_{2N}) = 0. \quad (\text{B33})$$

It is straightforward to show that only zero satisfies this system. This implies that  $F_{2N-1}$  is zero, a contradiction. (Or in terms of CFT,  $F_{2N-1}$  is a two-point function with a  $(2N-1)$ -leg boundary operator and a one-leg boundary operator (figure 14), and because these operators have different conformal weights, this two-point function is necessarily zero.) Thus, we have argued that if (B30) is true for  $i=1$  and  $s=1, \dots, 2N-2$ , then conjecture 14 is true. Interestingly,

our analysis also implies that conjecture 14 is true even if  $(x_{2N-1}, x_{2N})$  is not known to be a two-leg interval, although the conjecture is somewhat less elegant to state without this condition.

To finish, we use the techniques mentioned above to sketch a proof of conjecture 17 (in which  $\theta_1 = 0$ ). If we differentiate the three conformal Ward identities and the null-state PDEs centered on  $x_1, \dots, x_{2N}$  with respect to  $x_{2N+1}$  and let  $G := \partial_{2N+1} F$ , then we find the system of  $2N$  null-state PDEs,

$$\left[ \frac{\kappa}{4} \partial_i^2 + \sum_{j \neq i}^{2N+1} \frac{\partial_j}{x_j - x_i} - \frac{1}{(x_{2N+1} - x_i)^2} \right] G(x_1, \dots, x_{2N+1}) = 0, \quad i \in \{1, \dots, 2N\}, \quad (\text{B34})$$

together with the system of three conformal Ward identities:

$$\sum_{i=1}^{2N+1} \partial_i G(x_1, \dots, x_{2N+1}) = 0, \quad \left[ \sum_{i=1}^{2N+1} x_i \partial_i + 1 \right] G(x_1, \dots, x_{2N+1}) = 0, \quad \left[ \sum_{i=1}^{2N+1} x_i^2 \partial_i + 2x_{2N+1} \right] G(x_1, \dots, x_{2N+1}) = 0. \quad (\text{B35})$$

With respect to the function  $G$ , the coordinates  $x_1, \dots, x_{2N}$  still have conformal weight  $\theta_1 = 0$  while the anomalous coordinate  $x_{2N+1}$  has conformal weight one. The existence of the anomalous coordinate does not alter any of the statements in lemmas 3–4 and 18, and therefore  $G$  satisfies these lemmas too. In particular, if we suppose that  $G$  is not zero, then the arguments presented above imply that at least one interval among  $(x_1, x_2), (x_2, x_3), \dots, (x_{2N-2}, x_{2N-1}), (x_{2N-1}, x_{2N})$  is not a two-leg interval, and if we collapse it, then a straightforward adaptation of lemma 5 shows that we arrive with a nonzero function that is derivative of an element of  $\mathcal{S}_{N-1/2}$  with respect to  $x_{2N+1}$ . After we repeat the process until all intervals among  $(x_1, x_2), \dots, (x_{2N-1}, x_{2N})$  have been collapsed, we are left with a conformally covariant function that is not zero and depends only on the coordinate  $x_{2N+1}$  with conformal weight one. Because no such function exists, we conclude that  $G$  is zero.

The conclusion of the previous paragraph that  $G := \partial_{2N+1} F = 0$  may be inserted into the null-state PDE for  $F$  centered on  $x_{2N+1}$  to find

$$\sum_{j=1}^{2N} (x_{2N+1} - x_j)^{-1} \partial_j F(x_1, \dots, x_{2N}) = 0. \quad (\text{B36})$$

We fix the coordinates  $x_1, \dots, x_{2N}$  to arbitrary values, and we fix  $x_{2N+1}$  to  $2N$  distinct arbitrary values greater than  $x_{2N}$  in order to have an invertible system of  $2N$  equations in the unknowns  $\partial_1 F(x_1, \dots, x_{2N}), \dots, \partial_{2N} F(x_1, \dots, x_{2N})$ . The one unique solution of this system is the zero solution, so all of the unknowns equal zero. Finally, because  $x_1, \dots, x_{2N}$  were chosen arbitrarily, we conclude that  $\partial_1 F, \dots, \partial_{2N} F$  are zero too. Thus  $F$  must be a constant. This concludes our proposed proof of conjecture 17.

### Appendix C: Some comments about the bound (20)

We suspect that the solution space  $\mathcal{S}_N$  (definition 1) of the system of PDEs (10–11) comprises the entire solution space (i.e., space of all classical solutions (in the sense of [22])) of this system, deeming the bound (20) unnecessary. In this appendix, we give some reasons for why we believe this is true.

In the work we have presented, the growth bound (20) is mainly used in the conclusion of the proof of lemma 3 in the paragraph containing equation (53). In fact, one can envision ending this proof without using this bound if for each  $(\xi, x) \in \pi_{i+1}(\Omega_0)$ , we knew of a pointwise estimate such as

$$|\partial_j H(\xi; x, \delta)| \leq c(\xi, x) |H(\xi; x, \delta)|, \quad j \in \{1, \dots, 2N - 2\}, \quad 0 < \delta < b(\xi, x). \quad (\text{C1})$$

for some positive functions  $b$  and  $c$  (the latter bounded on all compact subsets of  $\pi_{i+1}(\Omega_0)$ ). Indeed, with such an estimate, (42) becomes

$$|H(\xi; x, \delta)| \leq C_1(\xi, x, b) + C_2(\xi, x, b) \int_{\delta}^b |H(\xi; x, \eta)| d\eta, \quad (\text{C2})$$

and an application of the Gronwall inequality (after substituting  $t = 1/\delta, s = 1/\eta$ ) shows that the supremum of  $|H(\xi; x, \delta)|$  over any compact subset of  $\pi_{i+1}(\Omega_0)$  is bounded as  $\delta \downarrow 0$ . The bound (C1) follows if we can prove, for example, that for each  $(\xi, x) \in \pi_{i+1}(\Omega_0)$ , there is a neighborhood  $\mathcal{U}_x \subset \pi_{i+1}(\Omega_0)$  of  $\xi$  and a  $b > 0$  such that

$H(\psi; x, \delta)$  is nonpositive or nonnegative for all  $\psi \in \mathcal{U}_x$  and all  $\delta \in (0, b)$ . In this event, the elliptic PDE (49) implies that  $H(\psi; x, \delta)$  satisfies the Harnack inequality [22]. That is,

$$\sup_{\psi \in \mathcal{U}_x} |H(\psi; x, \delta)| \leq C(\xi; x, \delta) \inf_{\psi \in \mathcal{U}_x} |H(\psi; x, \delta)| \leq \left( \sup_{0 < \delta < b} C(\xi; x, \delta) \right) |H(\xi; x, \delta)|. \quad (\text{C3})$$

The supremum of  $C$  over  $0 < \delta < b$  is finite because the coefficients of the PDE (49) are bounded over  $0 < \delta < b$ , and coupling this estimate with (50) gives the desired bound (C1). We therefore conclude that hypothetical solutions of the system of PDEs (10–11) satisfying this nonpositivity/nonnegativity condition satisfy the growth bound (34) of lemma 3. Therefore, a solution that violates this growth bound must both grow faster than a power law and violate this nonpositivity/nonnegativity condition as  $\delta \downarrow 0$ . For each  $(\xi, x) \in \mathcal{U}_x$ , such a solution would rapidly oscillate from positive to nonpositive or negative to nonnegative as its magnitude grows faster than a power law as  $\delta \downarrow 0$ . Because any solution to the system of PDEs (10–11) can serve as a  $\text{SLE}_\kappa$  partition function, we suspect that such “exotic” solutions do not exist, or else they could induce very irregular characteristics in the boundary arcs that are anchored to the endpoints brought together by the limit  $\delta \downarrow 0$ .

In fact, we can weaken (C1) by including a factor of  $\delta^{-1}$  on the right side and still obtain the results that would follow from (C1). This change introduces a factor of  $\delta^{-1}$  into the integrand in (C2), and then the Gronwall inequality shows that for each  $(\xi, x) \in \pi_{i+1}(\Omega_0)$ ,  $|H(\xi; x, \delta)| \leq c(\xi, x)\delta^{p(\xi, x)}$  as  $\delta \downarrow 0$ , where the supremum of the functions  $c$  and  $p$  are bounded over compact subsets of  $\pi_{i+1}(\Omega_0)$ . With this result replacing (20), the proof of lemma 3 can proceed exactly as presented in the paragraph containing equation (53).

So far, we have not been able to derive the estimates (C1) or similar estimates that would lead to similar, desired results without assuming other conditions. In the absence of such strong pointwise information, we have relied on combining the bound (20) that defines  $\mathcal{S}_N$  with the Schauder interior estimates from elliptic PDE theory to conclude the proof of lemma 3. This method seems unnatural because, in order to use it, we need to sum over the  $2N - 2$  PDEs in (43) to obtain the elliptic PDE (49) in the coordinates of  $\xi$ . In so doing, we discard a lot of the information contained in the original PDEs (43). Some of this discarded information could be useful for proving lemma 3 without the growth condition (20).

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